THE THEORY OF LIGHT

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THE THEORY OF LIGHT

A TREATISE ON PHYSICAL OPTICS

by

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IN THREE PARTS

PART I

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PREFACE.

THE aim of the treatise, of which this volume forms the first part, is to give an account of the theory of physical optics that is systematic, and as complete as possible within the somewhat narrow limits within which I have thought it expedient to confine myself. I hope, of course, that the work will be found useful to the student, and it is mainly in his interests that I have begun with a brief discussion of the aim and the method of science. I place no special value on my statement of this matter; but I do not think that it is easy to exaggerate the importance of understanding clearly what we are really aiming at. Without an agreement on this point we can have no criterion as to the propriety or otherwise of our methods, and no final concord as to our success or failure. And if I have marked out the limits of science in a form that frees it as much as possible from metaphysical difficulties, it is certainly not because I regard the "intrusion" of metaphysics into science as necessarily baneful. As a matter of fact every problem of science, in so far as it involves thinking at all, abuts on a problem of metaphysics, and we can scarcely be surprised if the workers in adjoining fields occasionally cross the boundary. There might be advantages in each keeping to his own domain; but if the physicist *will* be a trespasser, what he must be careful to avoid is not so much metaphysics as bad metaphysics. We can scarcely acquiesce in the simple faith of some men of science that their metaphysics is to be accepted merely because it is unconscious or naïve.

A serious difficulty in discussing the aims and methods of physics in a work that is mainly concerned with a special department of that science, is to keep the discussion within reasonable

limits. In a few pages one has to touch on questions that might easily occupy a book, and it may be that in my endeavour to avoid obscurity on the one hand and diffuseness on the other, I have "moved as in a strange diagonal" and satisfied no one. As everything is made to rest on a dynamical basis perhaps I should have subjected the fundamental concepts of that science to a careful scrutiny. This, however, would be more appropriate to a work on the Principles of Mechanics, and after some trial I have concluded that it could not be undertaken here without taking up more space than could be given to it. Perhaps, too, the student may be content to work with the tools that he has inherited, without denying that they would be the better for sharpening or that they might even be replaced by more modern articles. In any case, the writer has taken warning from his experiences in practical mechanics that an unskilled attempt to sharpen a tool is apt to remove whatever edge it has.

I hope that the foundation has been laid broadly enough to support the whole superstructure and that the mode of treatment will make clear the relation between light and electricity, when these two sciences come into contact. This will appear more fully in the second volume, which deals with those branches of our subject not discussed in this, viz. such matters as dispersion, the rotations—structural and magnetic—aberration, diffraction, etc. The connection between optical and electrical phenomena is often imperfectly presented. In some of the text-books it is scarcely referred to, except in a sketchy way towards the close. In others the science of optics is based on the electromagnetic theory in such a manner that students who have not made a careful study of the bases of electrical theory are ignorant of the real foundations. The natural method seems to be to lay such a dynamical foundation that all phenomena, electrical and optical, may be colligated with the aid of the same ethereal medium.

The third, and concluding, volume of this work will be devoted to a history of optical theories. Most modern works on light give at least a passing reference to the older theories, such as the elastic solid theory, sometimes even in such a way as to suggest that these theories are still tenable. It seems preferable to state the modern theory and, after working it out in detail, in order to

test its usefulness, to supplement this with a history of the past. We do this, not with the object of making dead bones live, but to get an insight into the manner in which a great scientific theory is actually built up, and to give a human interest to our study by learning something of the lives and modes of thought of the great men who have raised the structure.

A special feature of the work is the careful comparison between theory and experiment at every stage. From our standpoint as to the aim of science, such a comparison appears all-important. The one difficulty in this matter has been to get access to the most accurate and recent experimental results; a difficulty arising entirely from the present isolation of the writer. At the ends of the earth there are no great libraries to which to refer, and the procuring of material from 'home' is tedious and uncertain. In dealing with experimental results I have deliberately refrained from entering upon descriptions of the methods of experiment as, in my judgment, quite out of place in such a work as this.

A large part of this volume embodies in a modified form the substance of a series of papers of mine published within recent years by the Royal Society. I have rarely, however, made any direct reference to these or any other papers in the course of the volume. Such references, to be complete, would occupy a great deal of space, and the student who wishes to consult the original 'authorities,' as every serious student must, will obtain practically all the directions he can want in such a work of reference as Winkelmann's *Handbuch der Physik**.

My special thanks are due to Mr W. J. Harrison, of Clare, who has undertaken the revision of the proof-sheets, a task rendered impracticable by my residence at the antipodes.

R. C. M.

WELLINGTON.
June, 1907.

* However, at the suggestion of the Syndics of the Cambridge University Press, a few references have been supplied in footnotes by Mr Harrison.

CONTENTS.

CHAPTER I.

THE SCOPE AND METHOD OF THE INQUIRY.

"If, Theaetetus, you have a wish to have any more embryo thoughts, you will be all the better for the present investigation and, if not, you will be soberer and humbler and gentler to other men, not fancying that you know what you do not know."—PLATO.

"WHAT is the chief end of science?" should be the first question in the catechism of every physicist. We raise it here not so much for the reasons suggested by the words that Plato makes Socrates address to his pupil, although it is certainly well to know our limitations and to shake assurance from our creeds, as from the fact that in any inquiry the concepts that we employ and the methods that we pursue must be determined, wholly or in part, by the end we have in view.

What, then, is that end for physical science? Of the various answers that might be suggested we shall consider briefly the most important. Probably the most popular is that the end of science is to 'explain' the world, and our agreement or dissent must depend entirely on the meaning we attach to the word 'explanation.' It is a word used in many senses, but the one that will naturally occur to the mind of a person who has often been called upon to explain anything and has attempted to do so with any hope of success is *the reference of the unknown to the known.* An explanation seeks to point out resemblances between facts and to lay bare the connecting links between the unfamiliar and the familiar. It is the desire to achieve this end that makes the physicist so anxious to point out the traces that enable the Sun to drag a planet round its orbit. It is this, too, that sometimes urges him to declare that all physical action is by impact,

the idea of a collision and its consequences being so familiar. And the historical fact that the science of mechanics was developed before other branches of science has doubtless contributed to that purely mechanical view of the universe, whose general acceptance is, in the opinion of some, "that far-off divine event towards which the whole creation moves."

However, although science may often strive, and sometimes with success, to refer the unknown to the known, there can be no doubt that it much more frequently reverses the process and refers the known to the unknown. Innumerable instances will occur to the student, and of these a few will suffice. In dynamics we 'explain' most things by means of Newton's Laws of Motion, or by the conservation of energy, or the Principle of Action, or the law of gravitation. It is clear that the 'explanation' of so well known and familiar a phenomenon as the fall of an apple by the statement that "every particle in the universe attracts every other particle with a force that varies directly as the product of the masses of the particles and inversely as the square of the distance between them" cannot be properly described as a reference of the unknown to the known. Again in the theory of sound we 'explain' the music of our pianos by referring it to waves in the air, with the attendant obscurities of elasticity, of Boyle's law, Charles' law, the adiabatic law, and so forth. In heat, we 'explain' the heating of a poker by the vibrations of particles that no one knows by experience, or we account for the failure of an engine by the statement that "it is impossible for a self-acting machine, unaided by any external agency, to convey heat from one body to another at a higher temperature." Lastly in electricity and optics we 'explain' everything by reference to the ether, which, for aught anyone can say, may be a mere figment of the imagination.

It may, perhaps, be urged, that many of these are real 'explanations,' for they refer things that we do not understand to *laws* that are known. But how are these laws known? The best that can be said for them is that they are obtained by Induction from the facts of experience. What, then, is Induction? It was described by Bacon as a process by means of which we can obtain universal propositions from particular perceptions; but neither

Bacon nor anyone since him has succeeded in making it clear how this can be done by the method, or how, under any circumstances, we can get more out of a thing than we put into it. The actual process of building up a scientific law may be roughly described thus. The phenomena under consideration are regarded from particular aspects, as in all thinking, scientific or otherwise. In this way conceptions are formed which correspond in a partial and a symbolical manner to the phenomena. An endeavour is then made to connect these concepts by a law. There is no *logical* necessity for this, for it is conceivable that knowledge of universals might be denied us. At the same time we realise that, if there is to be any knowledge, there must be some law, so we *assume* that it exists and that it may be found. The fact that we are constrained to make this initial assumption is an interesting psychological fact, and it is a striking result that the consequences seem to justify the assumption. As to what the law is the process of induction can give us only hints. What is really done is to set out with some hypothesis and to compare its consequences with the observed phenomena; but the primary hypothesis cannot be found by rule and we can never through induction get the idea of necessity into the sequence. Agreement between the phenomena and the consequences of the hypothesis may make the hypothesis tenable, it can never prove it. Hence all reference to scientific 'laws' involve something that cannot be explained, so that, except by imputing a special meaning to the word, we cannot regard 'explanation' as the real end of science.

Another answer to our fundamental question is that the aim of science is to get rid of the arbitrary and the apparent and to reveal the real. The aim is certainly a lofty one and we have no *à priori* objections to the pursuit; for to say at the outset that the real is unknowable is to claim to know reality. Our objection is that if science is to attempt so difficult a task, it must devote itself to a much closer scrutiny of its methods and its concepts than is its wont, and it must enter the lists with the champions of different philosophical systems reasonably equipped for the fray. So far, however, those who have made the most serious contributions to science have seldom had either the time, the inclination, or the aptitude necessary for such work. Of recent years, at any

rate, the exhortation to "beware of metaphysics" seems to have had its effect, and on the rare occasions when leading men of science have disregarded it, the results have not been reassuring. The claim to find reality, if made at all, is usually put forward by the rank and file, which is less conscious of its limitations. These men find certain concepts, such as matter and energy, useful for the purposes of science. They spend their lives over problems in which these are the only concepts needed. They assume that no others can ever be needed. Next they take the extraordinary step of declaring dogmatically that these concepts are real things. "The only real things in the physical universe are matter and energy," say the authors of the *Unseen Universe*. This is worse than the conduct of the artist who, after devoting his life to portraiture, declared that the only real thing was paint. It takes certain qualities of things, arbitrarily abstracts something, and then declares the abstraction to be real and all that is real. This, of course, is to take a ghost for reality, and there is actually a school so frightened by the ghost as to worship it.

Some physicists who probably mean to convey the idea that the end of science is to grasp the real, express themselves in rather unintelligible language by assuring us that science reveals the nature of "things in themselves." However, the idea of a "thing in itself" proves, on examination, to be worthless, even if it be not meaningless. Some of the difficulties are suggested in the following words of von Helmholtz. "We speak of the solubility of a substance, meaning its behaviour towards water; we speak of its weight meaning its attraction to the earth; we may justly call a substance blue under the tacit assumption that we are speaking of its action upon a normal eye. But, if what we call a property always implies a relation between two things, then a property or quality can never depend upon the nature of one agent alone, but exists only in relation to and dependence on the nature of some second subject acted upon. Hence, there is really no sense in talking of properties of light that belong to it absolutely, independently of all other objects, and that are supposed to be representable in the sensations of the human eye. The notion of such properties is a contradiction in itself." If you agree with this, then when you have set aside all the qualities of a

thing, as necessarily involving a relation to something else, you have not much left to be worthy of a name. If, however, you insist on retaining the idea and the name of a "thing in itself," it is worthless for scientific purposes. For either it is related to phenomena or it is not. If it is not so related, it must be useless to the man of science who is interested in phenomena, *i.e.* in things as they appear to him. If, on the other hand, it is related to phenomena, you have merely doubled his difficulties. All the old problems of the phenomenal remain, with the added perplexity of their relation to this empty "thing in itself."

And if science has no interest in "things in themselves" and does not seek to reveal the real, it is equally true that it does not get rid of the arbitrary and the apparent. As a matter of fact the methods of science are often arbitrary in the extreme. Ordinary knowledge involves a certain measure of arbitrariness, as it selects what is of interest and dwells on certain aspects of things. Scientific knowledge carries the process a stage further, and this, not of course for any love of the arbitrary, but because it realises that to succeed it must limit itself. And so far from getting rid of appearances, these are what it specially strives to know and to master. The mysteries of any hidden world "behind phenomena" it is content to leave to others.

Let us turn to a third answer to the question with which we began. "The final aim of physical science," says von Helmholtz, "is to find the ultimate unchangeable causes of the processes in nature." Here, just as in the case of a search for reality, we have no *à priori* objections to raise against the quest. We do not, for example, refuse to look for causes because, as has sometimes been objected, the conception is 'anthropomorphic.' It is affirmed that the conception of cause arises from the thought of one being doing something to another and that this is 'anthropomorphic.' Our answer is that all concepts are necessarily so, for it is a truism, often strangely overlooked, that whether man be the measure of all things or not, he is certainly the measure of his own concepts. Our objection is that the conception of cause is a popular and loose one, and must be carefully scrutinised before it is admitted into the serious domain of science. Jones has met his death at the hands of Sikes. What is the cause of the murder?

"The depravity of the human heart" says the moralist, "the malice aforethought of Sikes" says the Crown Prosecutor, "the conversion of the potential energy of gunpowder into the kinetic energy of the bullet" says the physicist, "the rupture by the bullet of some vital organ" says the doctor. Each of these arbitrarily dwells on the special aspect that is interesting from his point of view and pronounces it the cause. The one point on which they all agree is that cause is not an "unchangeable thing," but is a condition. But which condition is to be described as the true cause? Here the man of science, seeking to avoid all arbitrariness, is placed in a dilemma. The concept includes too much or too little. The cause may mean *all* the conditions, in which case it embraces the universe and is the cause of everything that is, and not merely of the special event under consideration. If on the other hand, he attempts to narrow the idea and always to avoid the arbitrary, he has no alternative but to make the cause and the effect identical, and so practically to get rid of causation as a fruitful concept. It is this that has led him sometimes to drive out 'force' from mechanics, or to retain it only as a formal concept as an aid in the statement of a law. If, however, he wishes to retain causation, as he may well do, he must recognise that there is something arbitrary about it. It singles out some one of all the conditions of an event and speaks of this as the cause. The principle on which this selection is made must depend entirely on the end in view, so that cause is essentially an idea that he brings to the phenomena and not one that the phenomena impose on him.

And not only must it be admitted that there is something arbitrary in the relation between cause and effect, but the most important element in the relation as it is usually thought of and described must present great difficulties to the man of science. This is the element of *necessity* in the sequence. Whence comes this idea of necessity, does it exist in the objects or only in our minds? To say that it exists in the objects and to ascribe it, as is often done, to some "inner necessity of their development" leaves this inner necessity the greatest of unsolved problems; while to account for it by means of 'laws,' such as the uniformity of nature, requires the use of hypotheses that pass beyond

experience. If, on the other hand, we say that the idea of necessity exists only in the mind, we must be prepared to enter into investigations as to the nature and mode of working of the mind and so raise almost innumerable questions that do not belong to the sphere of physics as that science is ordinarily understood.

Thus far we have considered three different answers to the question with which we began. To explain the world, to reveal the real, or to show the causes of nature's processes, have each in turn been put forward as the end of science. Our chief objections are that the actual methods of science do not suggest any such end and that it is inexpedient to involve science in metaphysical and philosophical difficulties about which there is no immediate prospect of agreement. Science is interested only in experience; its end is to know and to communicate. At the outset experience is but an indefinite manifold, or a confused blur. Science, like all thought, seeks to introduce definiteness and order. It soon accumulates a mass of facts that no mind could grasp and no memory retain. Its aim is to find a principle or 'law' that will bind all harmoniously together. As soon as it realises this, it becomes a premeditated art, not only esthetic in its intensity, but consciously esthetic in its aim.

If we recognise this as a true description of the purpose of science, we shall admit that science is not much concerned with reality and that the solution of the various philosophical difficulties that have been raised is, for it, almost a matter of indifference. The practical end it has in view is served equally well whether its hypotheses correspond to reality or not. It may echo the words of Descartes "Même je crois qu'il est aussi utile pour la vie de connaître les causes ainsi imaginées que si on avait la connaissance des vraies," and perhaps even assent to the remark of Osiander in his preface to Kepler's books *De revolutionibus cœlestibus* "It is not necessary that scientific hypotheses be true or even probable; the one thing required of them is to reconcile calculation and observation." Of course no one reviewing the work of the great men of science can fail to be impressed with "the structure brave, the manifold music they build" and to ask in what sense it is real. No answer worth listening to can, however, be given without due consideration of the meaning of reality.

This is the province of the philosopher, and the man of science who invades it rashly is in danger of making an exhibition of his ignorance or of his incompetency for philosophic discussion. He is probably wiser to confine himself to phenomena and their laws, and to make no claim to account for everything. It is not for him to say that there is no pathway to reality, or that there is a fixed gulf between appearance and reality, or that it is not worth the attempt to bridge it. All he need say is that the search for reality and the ascertaining of ultimate truth are beyond his province. For such an one it is a great relief to be freed as much as possible from the incubus of metaphysics and to feel that he need not stay his progress to await the issue of the conflict between different schools of philosophy. Certain hypotheses for him are not so much false as useless.

According to the view here presented the aim of science is practical and esthetic. In its practical aspect it seeks to colligate experiences with the object of enabling us to know as much as possible and to communicate our knowledge. To achieve this great end it seeks a harmony, the contemplation of which gives it its esthetic interest. The harmony it expresses by a 'principle' or 'law.' If this be its aim, we should expect science to advance by trying hypothesis after hypothesis until it reaches one that fits in with all the facts. And whether this has been the conscious aim of science or not, there can be no doubt that this has been its method. This might be exemplified from every department of science and is strikingly illustrated in the field of optics—as the consideration of the history of our subject will show. A single instance from the more widely known field of mechanics may be taken. Let us recall the process by which Kepler obtained his laws of planetary motion. He began with the hypothesis that the distances of the planets from the Sun are determined by the six regular solids of geometry. "The Earth's orbit is the sphere, the measurer of all. Round it describe a dodecahedron; the circle including this will be (the orbit of) Mars. Round Mars describe a tetrahedron; the circle including this will be Jupiter. Describe a cube round Jupiter; the circle including this will be Saturn. Then describe in (the orbit of the) Earth an icosahedron; the circle described in it will be Venus. Describe an octahedron

round Venus; the circle inscribed in it will be Mercury." This law seems fanciful enough; but the only sound objection that could have been advanced against it at the time was that it did not harmonise well with the results of experiment. However, Kepler was at first well satisfied. He declared that "he would not barter the glory of the invention for the whole Electorate of Saxony." His ardour was somewhat cooled by Tycho's advice "first to lay a solid foundation for his views by actual observation." So he turned aside from these speculations, and next gave his mind to pondering over the *forms* of the planetary orbits. Naturally, he tried circles first, but Tycho's observations convinced him that, at least in the case of Mars, there was a considerable departure from the circular form. He therefore tried curves of various sorts, testing them by Tycho's observations, until he hit upon the ellipse. Thus he discovered his second law. His next step was to generalise this law by extending it to all the planets and he found it still true. This great success revived his interest in the problem that had first aroused his enthusiasm, that of the planetary distances. He tried hypotheses of all kinds. He compared the planetary distances with the intervals of the notes on the musical scale, an idea suggested by the venerable notion of the music of the spheres. At last he conceived the idea of comparing the powers of the different numbers that represent the distances of the planets with the powers of the numbers representing their periodic times. Thus he hit upon his third great law. "The die is cast—the book is written, to be read either now or by posterity, I care not which. It may well wait a century for a reader, as God has waited 6000 years for an interpreter of his works."

If we are agreed as to the end of science, we may then profitably discuss its method. In the field of optics we shall take it as our aim to bind together harmoniously the enormous mass of facts obtained by careful experiments on light, putting aside those whose discussion involves anything but the simplest physiological facts and theories. In any thinking about phenomena we must consider them under particular aspects, and speak of them in terms of certain concepts. It is all important that the concepts be clearly formed and that they be conceived in

the same way by those with whom we wish to communicate. Our choice of concepts is to a certain extent arbitrary; but it gives colour to the whole superstructure and determines the language in which our facts are to be described. The concepts that we shall employ most frequently are number, space, time, motion, substance, and energy. We shall assume that these have been cleared of contradictions as far as the nature of the mind will permit. To make any progress we must make certain postulates. We cannot begin with axioms and we certainly cannot echo Newton's statement "hypotheses non fingo," nor do otherwise than deny or explain away the remark with which he closes the Principia "Hypotheses seu metaphysicae, seu physicae, seu qualitatum occultarum, seu mechanicae, in philosophia experimentali locum non habent."

In the first place we postulate the existence of the ether, an all pervading medium through which what we call waves of light may be propagated. This medium is literally metaphysical; but it is a substance that has some of the qualities of ordinary matter. Indeed it could not be so much as thought of, if it had not, since our conception of substance is necessarily derived from our sensations. It is an abstraction obtained by selecting some of the qualities from media, such as water or jelly, with which we are familiar, and rejecting other of the qualities. Our excuse for postulating such a medium is that we see no prospect of attaining our end without it, while with it great things can be accomplished, not only in the domain of optics, but in the fields of electricity and magnetism as well. At the same time we need not concern ourselves with the 'reality' of the ether, nor take the unphilosophic step of pronouncing it to be real merely because it is a conception that is convenient for our special purpose.

Our next postulate, or set of postulates, comes in when establishing a correspondence between our concepts and what is given us in experience. We want, for example, to think of a relation between the frequency of waves and our sensation of colour, or between the energy associated with a certain portion of the ether and our sense of the brightness of a light. To attempt to establish such a relation in a thorough fashion would take us into the depths of philosophical difficulties. We must be content

to postulate the correspondence, to say, for example, let it be granted that a certain frequency corresponds to the sensation of 'blue' in a normal eye. This is no doubt arbitrary; but it is part of our art to suggest the correspondence that makes our theory fit in with the facts.

The last postulate is made in laying down the fundamental 'law' or dynamical principle that is to bind all our concepts together. What should be the characteristics of this law? We need scarcely say that it should fit in with the facts, for this is the whole end of it. It should suggest all the experiences that we have and none that we have not. We want it to be logical, that is not self-contradictory, and it is desirable that it should not contradict "well established" principles in other departments of thought. Then it should be as simple as possible. This we want for convenience and lucidity, if not from a conviction of the truth of Newton's comment on the first of his *Regulae Philosophandi*: "Natura enim simplex est et rerum causis superfluis non luxuriet." We want, then, few concepts and the simplest relation between them. Finally, we want a principle that will present the facts as vividly as possible.

We should scarcely be surprised if we find it difficult to satisfy all these demands, and if men differ as to which is the most important. Some would sacrifice simplicity to vividness, others will have simplicity at all costs. Thus some, in their anxiety to work with the fewest concepts, drive force and energy out of mechanics except as formal concepts to be defined in terms of mass and motion, concealed or otherwise. Others think that even if this does not render the definitions of mass and force 'circular,' it abstracts so much as to leave mechanics almost ghostly. It would be out of place to enter into the controversy here. We shall make constant use of energy as a concept derived from our experience of being able to exert ourselves and do work; whether this concept can be resolved into "simpler" ones or not will not in any way affect the validity of what we do. Nor need we delay our progress to decide the question whether it is possible or expedient to regard all potential energy as the kinetic energy of concealed masses,' for whether this be so or not will merely affect the interpretation of certain terms in our formulae.

The conflict between vividness and simplicity is also raised by the question as to the part that the use of models should play in the development of a scientific theory. Different views on this question will be taken by men of different types of mind. One can seize a large number of facts vividly, the other needs a general idea to coordinate them all. Of these types of mind, the one makes the engineer, the 'practical' man, the man of business or of affairs, the other makes the philosopher. The first will delight in models to 'explain' his theory, the second will rest more content in contemplating an all embracing law. Thus we find von Helmholtz confessing that he is more at home with a differential equation than with a model, while Lord Kelvin almost goes the length of saying that we know only what we can construct. "I never satisfy myself," he says, "until I can make a mechanical model of a thing. If I can make a mechanical model I can understand it. As long as I cannot make a mechanical model all the way through I cannot understand." This is one of the many interesting personal touches in the famous *Baltimore Lectures*, and for these we are grateful as giving us an insight into the workings of a mind that has done so much to enrich our science. We are, however, less grateful when the lecturer appears to suggest that *his* mode of thinking is the only possible one, and sets up the question "Can we make a mechanical model of it?" as the test whether we do or do not understand any subject in physics. The usefulness of models no one seriously questions. They are often a great aid to the scientific imagination in suggesting analogies between the unfamiliar and the familiar, and they may add greatly to the vividness of our conceptions. In this way they are particularly useful for purposes of exposition, when we are trying to convey to others a conception that they have not yet clearly formed. But their limitations and dangers are almost too obvious to mention. At the best they present partial analogies, and illuminate only special portions of the theory. No one would suggest the possibility of constructing a model that would represent *all* the phenomena of light and electricity, whereas it is not only possible but probable that a single dynamical principle will be found to comprehend them all. The dangers of a model are that it is apt to suggest false analogies as well as true ones,

that it is difficult to separate the truth from the falsehood, and that it may easily give a groundless sense of security by suggesting 'common sense' notions, that is inductions from experience that have not been rigorously scrutinised.

It must be clear, then, that if we are to fulfil the purpose of science as we have described it, we cannot rely entirely on models, but must use them only as convenient illustrations of special aspects of some general dynamical principle. But how is this principle to be obtained? There is no royal road to it; it can be reached only by the patient and laborious process of trial and error, of setting out with hypotheses and seeing how their consequences fit in with the facts. Newton insisted that the only hypotheses to be admitted are those suggested by induction from experience: "quicquid enim ex phaenomenis non deducitur, hypothesis vocanda est; et hypotheses in philosophia experimentali locum non habent." To this, however, we must demur. The hypothesis is the same however it is reached, and it may be welcomed from any source. No doubt we should expect greater things from hypotheses that have been obtained by generalisations from experience, for we know at the outset that they accord with some of the facts. But the golden fruits of knowledge often ripen where they are least expected, and a study of the history of science will soon convince us that the process of induction as described in the text-books of logic, has done far less to suggest the leading principles than might be supposed.

Fortunately at this stage in the development of mechanics we have not to go far in search of general principles. At first sight our difficulty seems rather an *embarras des richesses*; we must make a selection from different principles each of which may claim some special advantage. In making our choice, we must, of course, be guided by the special end we have in view, and the selection then proves a simple matter. Three different courses seem open to us. We might take as our fundamental principle or principles, the Laws of Newton combined with d'Alembert's Principle. These have the advantage of having been long in use, and so of being thoroughly familiar to the student. However, even if the doubts and difficulties with which they are involved be removed, they are not convenient for the special problems of optics. We have to

deal with a machine the inner workings of which are hidden and do not concern us. To discuss its action by means of Newton's Laws we should have to make various hypotheses as to the nature of the mechanism. This would involve us in further doubts and difficulties, and even if we had overcome them to our satisfaction, we should not have obtained a formula more convenient for purposes of mathematical development, than that derived from the Principle of Least Action, which in this book will be taken as the fundamental dynamical principle.

Another method of procedure would be to combine the Principle of Inertia with Gauss's Principle of Least Constraint, and, following Hertz in his *Principles of Mechanics*, formulate the "Law of the Straightest Path" as the basis of dynamics. One objection to this is that we should have to dress up our theory in a somewhat unfamiliar garb. Another is that Hertz commits us to the dogma that all potential energy is really the kinetic energy of concealed masses in motion. This is an idea made familiar by the speculations of physicists such as Lord Kelvin and von Helmholtz; but it seems inexpedient to make this hypothesis the very basis of all our reasoning. At the same time it should be noted that the Principle of Action that we are to adopt can be deduced, with proper assumptions, from Hertz's Principle, and also from the more classical Laws of Newton, so that anyone who prefers to build on these foundations can soon reach a stage at which our analysis becomes intelligible.

Our procedure will be to postulate the Principle of Least or Stationary Action. It is a principle that seems wide enough to comprehend the whole of mechanical science, including all physics in that term*. The only serious doubt as to this is the criticism of Hertz†, that the principle cannot be applied to the motion of rigid bodies rolling on one another. Larmor has indicated‡ how the objection may be removed; but even if we regard Hertz's criticism as fatal to the Law of Action as the basis of *all* mechanics, we may still admit the principle within the field of optics and

* See von Helmholtz: *Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung*. Wiss. Abh. III. 203; and Larmor: *Aether and Matter*. Appendix B.

† *The Principles of Mechanics* (trans. Jones and Walley), p. 19.

‡ *Aether and Matter*, p. 277.

electricity, with which alone we are at present concerned. In that field we have nothing to do with the rolling of rigid bodies and the course of our investigation will determine whether the principle is wide enough for all the facts or not.

The Principle of Action has been presented in various forms since its first hazy enunciation, a century and a half ago, by Maupertius*. The form that we shall find most convenient is that which makes everything depend on the variation of a Principal Function, involving the difference between the kinetic energy and the potential energy of the system in any configuration. The principle states that the natural course of the motion from one configuration A to another B is one that makes the time integral of $T-W$ (the difference between the kinetic energy and the potential energy) stationary as regards such slight variations of the path from A to B as keep the time of passage unvaried. Of course if the term 'potential energy' is to have any definite meaning the work that the system can do must depend only on the actual configuration and not on the path by which it has been reached, so that the principle, as thus stated, is applicable only to conservative systems. It may, however, be extended to non-conservative systems by including in the variation the work of any forces that are not involved in the potential energy. The virtual work of these forces added to the variation of the Principal Function must then be made to vanish. The only limitation on the application of the principle now is that the coordinates must be really independent, and that the configuration of the system must be completely specified in terms of them without including their differential coefficients with respect to the time.

Postulating this dynamical principle our aim is to show by means of it that all the varied and complex phenomena of physical optics may be woven together harmoniously by regarding them as due to periodic disturbances in a medium that we call the ether.

* See von Helmholtz: *Zur Geschichte des Princips der kleinsten Achin.* Wiss. Abh. III. 249.

CHAPTER II.

PRELIMINARY IDEAS.

It has been said that we are to regard the phenomena of light as due to periodic disturbances propagated in a medium that we call the ether. The essential feature of a periodic disturbance is that the characteristics of the motion are all repeated after a definite interval of time which is called the period. One of the best known examples of such a motion is that afforded by the progressive waves set up in the surface of a sheet of water by some disturbance, and it is due to the analogy with such waves that the movements of the ether discussed in optics are called waves and the modern theory of light is spoken of as the wave theory. While the wave presents the most familiar example of periodic motion to the ordinary man, the type of this motion that the student of science may be expected to know best is Simple Harmonic Motion. In this case the displacement of a moving point about a fixed centre is represented by an expression of the form $\eta = a \cos p(t+\epsilon)$, where a is the amplitude, $2\pi/p$ the period, $p/2\pi$ the frequency, and ϵ the phase of the vibration expressed in time. A displacement closely related to this and one that is more important for our purposes is that represented by $\eta = a \cos p(t+\epsilon - x/v)$.

For a given value of x the motion is Simple Harmonic, so that if η be at right angles to x, the point moves up and down at right angles to the axis of x. Let the positions of $P_1, P_2, \ldots P_n$ in Fig. 1 indicate the displacements at any time at different points on the axis of x, the curve drawn through P_1, $P_2 \ldots$ being the well-known sine or cosine curve. At a later time the displacements will be indicated similarly by $P_1', P_2', \ldots P_n'$, lying on another

sine or cosine curve. It thus appears that $\eta = a \cos p(t + \epsilon - x/v)$ represents a wave form with a sine or cosine curve for its outline moving with velocity v along the positive direction of the axis of x. At any given moment η and $\frac{d\eta}{dt}$ have equal values at points for which px/v differs by 2π, i.e. when x differs by $2\pi v/p$. This quantity is consequently called the wave length and is usually denoted by λ, so that we may write the displacement in the form $\eta = a \cos 2\pi/\lambda . [vt - x + \epsilon_1]$, where ϵ_1 is the phase expressed as a length.

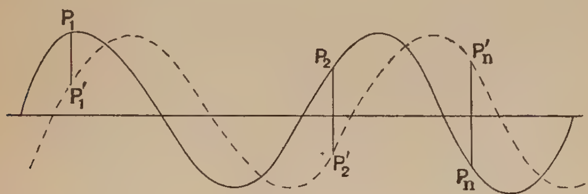

Fig. 1.

More generally if $y = f(x)$ is the equation of any curve, it is obvious that $y = f(x - vt)$, where v is a constant, represents a wave having a curve of this form for its outline, advancing with velocity v along the positive direction of the axis of x. The consideration of this more general type is readily reduced to that of the simpler one dealt with above, by means of Fourier's theorem. According to this theorem any function of x and t of the kind that presents itself when we are dealing with optical problems connected with plane waves can be expressed as a series of the form $\sum_1^\infty a_n \cos n . 2\pi/\lambda . (vt - x + \epsilon_n)$, where v is a constant. It follows from this that the consideration of the simpler type of wave represented by a circular function will enable us to deal with more complicated cases when they arise. As any periodic function may be resolved into constituents of the simpler type, we may regard a source of light as a centre from which waves of this type emanate. If f be the frequency, v the velocity of propagation of the wave, and λ the wave length, we have $\lambda = v/f$, so that if v is a constant the wave length is inversely proportional to the frequency. We shall take it that the colour depends on the

frequency and therefore, in a given medium, on the wave length, so that if all the waves are of the same length the light will be monochromatic. Although it may be difficult, if not impossible, to produce such light in actual practice, it will be none the less convenient to begin by investigating the characteristics of its propagation and consider when necessary the modifications that must be introduced when the light is no longer of this simple character.

The waves that we have been considering are plane waves, the front of the advancing waves being always a plane perpendicular to the axis of x. In attempting to solve the optical problems actually presented in nature, it will not always be possible to confine ourselves to waves of this very simple type. Thus if we are dealing with a disturbance spreading out from a centre it may be necessary to take account of the different directions in which the waves are propagated. However, the same general principles as before will enable us to deal with such a case. We shall merely have to suitably modify the system of coordinates employed, and although this may complicate the analysis, it will not otherwise modify the procedure.

Returning then to the case of a wave of simple type in which the displacement is given by $\eta = a \cos 2\pi/\lambda . [vt - x + \epsilon]$ we see that there are four constants required to specify such a wave. These four are the wave length $\lambda = v/f$, the phase ϵ, the velocity of propagation v, and the amplitude a. Our dynamical principles will, as we shall see later, enable us to determine the magnitude of v in any given case, but the other three constants may have arbitrary values. The phase will depend on the choice of origins for x and t, the frequency f on the colour of the light employed, and we shall find that the amplitude is intimately related to the intensity. The intensity of a beam of light might be measured experimentally in various ways, e.g. by its physiological action, by its chemical or photographic action, or by its heating effect. These three measures will not usually be the same, so that in speaking of intensity we must understand with which of these intensities we are dealing. The photographic action is in some circumstances the one most capable of exact measurement, and if we are dealing with this we may say that two beams of light are of equal intensity when they affect similar photographic plates

similarly. This gives us an experimental means of comparing the intensities of two beams and so of measuring any one in terms of some unit. When, however, we come to apply our theory to the results of experiment we are at once faced with the question—what quantity or quantities connected with a wave in the ether may be taken as measuring the intensity? As all the actions referred to above, photographic or other, involve the expenditure of energy it is clear that the intensity must be connected in some way with the energy in the neighbourhood of the point considered; but the exact relation between the two cannot be settled *à priori*. It is plain, however, that in estimating the intensity we are not concerned with the energy at any definite moment of time, but with the mean energy over a large number of periods. The period varies with the colour, but the variation is not great within the limits of the visible spectrum where the period is always of the order 10^{-15} of a second. The impression of light on the retina persists for about one-tenth of a second, so that if we measure the intensity of the beam by its physiological action an enormous number of periods must elapse before there is any measurable effect. This will be true also for the most rapid photographic plate, so that the intensity must depend on the energy transmitted across a surface, or the energy per unit volume, averaged over a large number of periods or on what may be called the mean energy. We have still to determine whether the intensity is measured by the mean energy, or the mean potential energy, or the mean kinetic energy. Which, if any, of these is to be taken as corresponding to the facts can be settled only by comparing the results of the optical theory with those of experiment. This will be undertaken in due course; but it may be noted here that very few of the numerous estimates of intensity that have been made throw any light upon the question. These experiments nearly all deal with sensibly plane waves moving in the same direction in an isotropic medium and the beams compared are in the same medium such as air. Under these circumstances we shall prove later that the potential energy is proportional to the kinetic, and so also to the total energy, and hence all the measures of intensity suggested above will lead to the same result.

We have been concerned so far with a single disturbance setting out from some source, but we may have to deal with different disturbances proceeding from the same or from different sources. As the actual displacements in the ether postulated by our theory are extremely small according to any ordinary standard, it will be natural to apply the general principle of the superposition of small displacements to a problem such as that just suggested. This principle directs us to consider the displacement due to each disturbance separately and obtain the total displacement by algebraic addition of the several parts. It is, of course, conceivable that one disturbance might so alter the character of another that this simple method of superposition would not be legitimate, but it is reasonable to adopt the principle as a working hypothesis which will be amply justified by the agreement of the results with the facts of experience. The principle is usually spoken of in optics as the Principle of Interference, an ill-chosen term that has, however, become stereotyped in works on the theory of light.

One of the first applications of the principle of superposition in the domain of optics is the derivation of Huyghens' Principle, which plays a leading part in most expositions of the theory of light. If light be radiating from a source O the medium around this point will be disturbed and the discussion of the characteristics of the disturbance is facilitated by considering the form and position of a certain surface called the *Wave Surface*. This surface we may define most conveniently for present purposes as the locus of points at which the disturbances are in the same phase. Its form will depend on the nature of the medium. In the simple case of an isotropic medium, i.e. a medium whose features are the same in all directions, it is obvious from symmetry that the wave surfaces are spheres with O as centre. In other media, however, the form of the wave surface is less obvious, and it will be the subject of investigation later. Let us consider a disturbance spreading out from a source O and so giving rise to a wave surface S. Huyghens' principle asserts that in considering the further propagation of this disturbance we may disregard the origin of the disturbance O and fix our attention on what is taking place at S, that we may look upon each point of S as a new

centre of disturbance provided that we arrange that the various secondary waves proceeding from all the points of S, when combined in accordance with the Principle of Superposition, produce the same displacement at any subsequent time as that which would have been produced by the original disturbance emanating from O. The analytical expression and critical discussion of this principle will be more appropriately undertaken at a later stage, if not as a matter of logical arrangement, at any rate on the grounds of simplicity of treatment at the outset. Meanwhile it will be assumed that the reader is familiar with some of the simpler deductions from the principle to be found in the most elementary works on the theory of light from the descriptive side. Thus, e.g. if S be the wave surface at time t and S' at time t', then S' is the envelope of the secondary waves setting out from the various points of S. If, moreover, the secondary wave from A touches S' at A', then the disturbance at A' due to all the secondary waves from the various points of S, *except those in the immediate neighbourhood of* A, is null, the different displacements neutralizing one another when superposed. Thus the points A and A' may be spoken of as corresponding points, for as far as the effective disturbance is concerned the displacement at A' depends only on that at A.

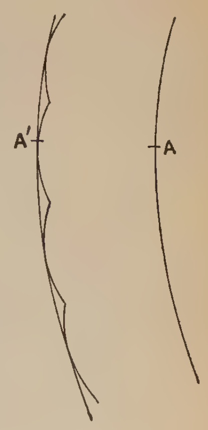

Fig. 2.

Thus far we have spoken of a wave, and this is the fundamental idea in the theory of light from the modern point of view. There is, however, another idea that has played a very important part in the development of the theory and is still convenient as a subsidiary idea. This is what is called a ray. The idea of a ray may be conceived in two different ways. From the first point of view let AB be a wave front moving parallel to itself so as afterwards to take up the position $A'B'$. The disturbance due to an element near A affects an element near A', its effect at other points of $A'B'$ being neutralized by the influence of the secondary waves due to the other elements. Thus the effective disturbance travels along AA' and this direction of propagation of a labelled

disturbance is called a ray. The velocity with which the disturbance travels along the ray is called the ray velocity, and it is obvious that the velocity with which the wave is propagated is equal to the component of the ray velocity along the wave normal.

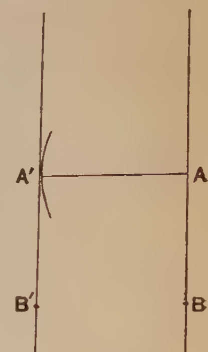

Fig. 3.

The second outlook is somewhat different. From this we have regard to the energy flowing from A as a source of light. If a screen S' were interposed between A and $A'B'$ it would prevent the energy that flows out of A from reaching every part of the surface $A'B'$. There would thus be a part of this surface that no energy would reach. This part would not be illuminated, nor if sensitive to light, like a photographic plate, would it be affected. We could therefore regard the energy as flowing out of A along certain lines, such as AA', and these lines of flow of energy we may call rays. Thus a ray from A to A' might be defined as the locus of the points at which an opaque body must be placed to cut off the influence of the light at A'. Of the two points of view that we have indicated the second is the one more commonly taken by the physicist; but it will appear later that the ray is the same from whichever of these points of view it is regarded.

The problem of determining the directions of the reflected and refracted rays corresponding to a given incident ray will be dealt with later; but a theorem, due to Huyghens*, that is intimately connected with this problem will be most conveniently given here. In stating this theorem it will make for brevity if we represent planes and lines by their intersections with the plane of the paper. Let OT be the interface between two media with different optical properties and OR a wave front at any time t. As this wave advances it will give rise to disturbances along OT, from each point of which secondary waves will emanate in accordance with the principles explained above. After a unit interval, i.e. at time $t+1$, the secondary waves will all touch a plane TQ where T is the line of intersection of the interface and the wave front in the first medium at time $t+1$. This plane TQ is the refracted

* Huyghens: *Œuvres* IX. p. 382.

wave and the corresponding ray is the line OQ where Q is the point of contact of the plane TQ with the wave surface whose centre is O, for O and Q are corresponding points in the sense mentioned above, since Q is the point where the secondary wave from O touches the wave surface at time $t+1$. If the lower medium had been the same as the upper we could, by Huyghens'

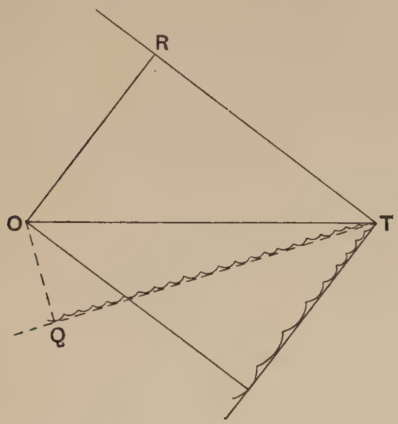

Fig. 4.

principle, have found the disturbance at time $t+1$ by considering the envelope of the secondary waves emanating from the various points of OT. This envelope would be a plane TA through the line T, a plane representing the position of the incident wave at time $t+1$ had there been no change of medium. Hence if we draw the wave surface characteristic of the upper medium and having O as centre, the incident ray is OA where A is the point of contact of the plane TA with this wave surface. Thus we have the following construction for the refracted ray corresponding to a given incident ray. Let OA be the direction of the incident ray, O a point in the interface OT. With centre O draw the wave surface for unit interval of time characteristic of the upper medium and let OA cut this in A. Draw the tangent plane to the wave surface at A and let this cut the interface in the line T. Through this line draw a tangent plane to the wave surface for unit interval of time characteristic of the lower medium and let the point of contact be Q. Then OQ is the refracted ray. Of course if the wave surface consists of more than one sheet this

construction will apply to each of the sheets separately and we shall have more than one refracted ray. The construction thus described is called Huyghens' construction. It may, if desired, be replaced by a modification due to Hamilton whose construction is derived from Huyghens' by reciprocating with respect to a unit sphere of which O is the centre.

So far we have said nothing of the displacement at any point except that it is a periodic function of the time which, by means of Fourier's theorem, may be resolved into a series of circular functions. In the most general case the displacement at any point (x, y, z) may be resolved into its components (ξ, η, ζ). Each of these components may then be expressed as Fourier series, so that we shall have $\xi = f_1(x, y, z, t)$, $\eta = f_2(x, y, z, t)$, $\zeta = f_3(x, y, z, t)$. If these functions are quite independent, there will be no relation between ξ, η, ζ, so that we cannot find a definite orbit for any point of the medium by eliminating x, y, z and t from the equations for ξ, η, ζ. However in many cases that actually present themselves the relations are such that this elimination can be performed. Under such circumstances each point of the medium on a given wave surface is describing a definite orbit, and the light is then said to be polarised. If the orbit is a straight line, a circle, or an ellipse, the light is said to be rectilinearly, circularly, or elliptically polarised as the case may be, although rectilinear polarisation is more commonly spoken of as plane polarisation.

We have seen that a wave of simple type where the displacement is represented by a circular function practically includes all cases that arise, and as any wave surface may be regarded as the envelope of its tangent planes we may substitute a system of plane waves for a wave of any form. Thus the consideration of a displacement whose components are given by

$$\xi = u \cos 2\pi/\lambda \,.\, [vt + \epsilon_1 - (lx + my + nz)];$$

$$\eta = v \cos 2\pi/\lambda \,.\, [vt + \epsilon_2 - (lx + my + nz)];$$

$$\zeta = w \cos 2\pi/\lambda \,.\, [vt + \epsilon_3 - (lx + my + nz)];$$

which for brevity we may denote by

$$(\xi, \eta, \zeta) = (u, v, w) \cos 2\pi/\lambda \, [vt + \epsilon - (lx + my + nz)],$$

will lead us to results of a far reaching character, although such a displacement is primarily characteristic of a plane wave of simple

type, moving with velocity v along a line whose direction cosines are l, m, n. The labour of discussing such a displacement is greatly lightened by replacing the circular functions by their exponential equivalents and employing a kind of shorthand in which (ξ, η, ζ) is written in the form

$$(u, v, w)\, e^{i\,2\pi/\lambda\,[vt\,-\,(lx+my+nz)]} = (u, v, w) e^{i\theta}$$

say, it being understood that as ξ, η, ζ are necessarily real, only the real parts of the complex quantities are to be retained. With this convention we see that if u, v, w are real and equal to $(\lambda, \mu, \nu)A$ we have $\xi = \lambda A \cos\theta$, $\eta = \mu A \cos\theta$, $\zeta = \nu A \cos\theta$ so that $\frac{\xi}{\lambda} = \frac{\eta}{\mu} = \frac{\zeta}{\nu}$. Hence the displacement is always directed along a fixed straight line and the light is polarised rectilinearly. If, however, u, v, w are not real we may put

$$(u, v, w) = (\lambda + i\lambda', \mu + i\mu', \nu + i\nu')A$$

and we then have $\xi = A(\lambda\cos\theta - \lambda'\sin\theta)$, $\eta = A(\mu\cos\theta - \mu'\sin\theta)$, $\zeta = A(\nu\cos\theta - \nu'\sin\theta)$. Solving any two of these for $\sin\theta$ and $\cos\theta$ and eliminating θ we get an equation representing an elliptic cylinder. If, further, as is always the case in optical problems, we have $l\lambda + m\mu + n\nu = 0$ and $l\lambda' + m\mu' + n\nu' = 0$, it follows that $l\xi + m\eta + n\zeta = 0$. Hence the point (ξ, η, ζ) moves on a plane section of an elliptic cylinder, so that its orbit is an ellipse. In this case the light is said to be elliptically polarised, and of this type circular polarisation is only a special case.

We have now to discuss some special examples of these different kinds of polarisations and to consider more particularly the effect of superposing two differently polarised beams. The simplest case is that of two rectilinear vibrations in the same direction. If this direction be taken as the axis of x we may represent the two displacements by expressions of the form $\xi_1 = r_1 e^{i\theta_1}$ and $\xi_2 = r_2 e^{i\theta_2}$. Any complex such as $\xi = re^{i\theta}$ may be represented in the well-known manner by a straight line of length r making an angle θ with some fixed line. With this geometrical symbolism the length of a line corresponds to the amplitude of the vibration and the angle between two lines to the difference of phase between the corresponding vibrations. If, then, we wish to superpose two different vibrations of this simple type, we have

merely to compound two vectors, such as OP_1 and OP_2, by means of the ordinary law for compounding vectors, i.e. by the parallelogram law. A formula for the resultant amplitude and phase is readily obtained from this geometrical construction. Thus if the components be represented by $\xi_1 = a_1 e^{i(pt+\epsilon_1)}$ and $\xi_2 = a_2 e^{i(pt+\epsilon_2)}$, and the resultant by $\xi = a e^{i(pt+\epsilon_1)}$ we have

$$\tan \epsilon = \frac{a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2}{a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2}$$

and $a^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\epsilon_1 - \epsilon_2) = a_1^2 + a_2^2 + 2a_1 a_2 \cos \Delta$, where Δ is the difference of phase between the two components. Three special cases of this formula for the amplitude of the resultant vibration are of frequent occurrence. When the difference of phase is zero or an even multiple of π we have $a = a_1 + a_2$; when this difference is an odd multiple of π we have $a = a_1 - a_2$; and when $\Delta = \pi/2$, or any odd multiple thereof, we have $a^2 = a_1^2 + a_2^2$. We shall prove later that the intensity of the light is proportional to the square of the amplitude of the vibration, so that if $\Delta = \pi/2$ the intensity of the resultant beam is equal to the sum of the intensities of the components. Vibrations for which this is true are called conjugates.

If we wish to compound several vibrations we have merely to make repeated applications of the parallelogram law. Thus, if there are n components whose amplitudes are represented by $OP_1, OP_2, \ldots OP_n$ and phases by the angles $P_1OX, P_2OX, \ldots P_nOX$ where $P_nOX = \epsilon_n$; then the amplitude of the resultant vibration is $n.OG$, where G is the centre of mean position of the points $P_1, P_2, \ldots P_n$, and the phase of the resultant is measured by the angle GOX. The same result may be expressed in another geometrical form, which is more convenient for some purposes. To obtain this we draw OP_1 equal to a_1 and making an angle ϵ_1 with a fixed line OX, P_1P_2 equal to a_2 and making an angle ϵ_2 with OX and so on. Then OP_n represents the magnitude of the resultant vibration and the angle

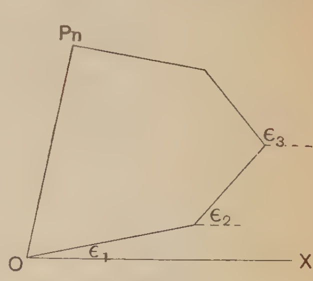

Fig. 5.

$P_n OX$ its phase. In the special case when the component vibrations form a series in which the amplitudes vary continuously and the phase difference between consecutive members is infinitesimal, the part of the polygon $OP_1P_2 \ldots P_n$ becomes a continuous curve. The element of length at any point P of this curve represents the amplitude of the vibration in the corresponding constituent, while the angle that the tangent at P makes with OX represents the phase of this constituent.

Turning from the case of displacements all in the same direction, we shall next consider the effect of compounding two vibrations that are at right angles to one another—e.g. along the axes of x and y respectively. The displacements in such a case can be represented by $\xi = ae^{ipt}$ and $\eta = be^{i(pt+\Delta)}$, where a and b are the amplitudes of the vibrations and Δ is the difference of phase between them. Confining ourselves to real quantities we have $\xi = a \cos pt$ and $\eta = b \cos (pt + \Delta)$, and on eliminating t between these equations we get

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} - \frac{2\xi\eta \cos \Delta}{ab} = \sin^2 \Delta.$$

The orbit in the resultant vibration is therefore an ellipse and the light is elliptically polarised. The elements of the elliptic orbit are obtained from the above equation by means of the formulae investigated in the discussion of Conic Sections. In particular the azimuth (ψ) of the major axis of the ellipse is given by the equation

$$\tan 2\psi = \frac{2h'}{a' - b'} = \frac{2ab \cos \Delta}{a^2 - b^2}.$$

In the special case when $\Delta = \pi/2$, we have $\psi = 0$, and the axes of the elliptic vibration coincide with the directions of the component vibrations. For some purposes it is important to decide the sense of the oscillation, i.e. to know whether the elliptic orbit is described clockwise or counter-clockwise as viewed from the positive side of the plane $z = 0$. This is easily determined by considering the signs of ξ, η, $\frac{d\xi}{dt}$ and $\frac{d\eta}{dt}$ at any time. Thus, if a, b, p are all positive and $a > b$, the motion is clockwise if Δ lies between O and π, and counter-clockwise if Δ lies between π and 2π. A change of the sign of p necessitates the reversal of this statement.

The elliptic orbit may degenerate into a straight line or it may become a circle. The degeneration occurs if $\sin\Delta=0$, or if $\Delta=n\pi$. If Δ is an even multiple of π the ellipse becomes the straight line $\frac{\xi}{a}-\frac{\eta}{b}=0$, while if Δ is an odd multiple of π the straight line is $\frac{\xi}{a}+\frac{\eta}{b}=0$. The conditions for circular polarisation are

$$\Delta=(2n+1)\frac{\pi}{2} \text{ and } a=b.$$

The components of the vibration are then $\xi=a\cos pt$ and $\eta=a\sin pt$, and the motion is clockwise or counter-clockwise according as p is negative or positive. It is convenient occasionally to replace a rectilinear or elliptic vibration by a pair of circular vibrations. Thus we may replace $\xi=a\cos pt$ by $\xi_1=\frac{a}{2}\cos pt$, $\eta_1=\frac{a}{2}\sin pt$ and $\xi_1=\frac{a}{2}\cos pt$, $\eta_2=-\frac{a}{2}\sin pt=\frac{a}{2}\sin(-p)t$. The first corresponds to a counter-clockwise and the second to a clockwise circular motion. Similarly we may replace the elliptic vibration $\xi=a\cos pt$, $\eta=b\sin pt$ by the circular pair $\xi_1=\frac{a+b}{2}\cos pt$, $\eta_1=\frac{a+b}{2}\sin pt$ and $\xi_2=\frac{a-b}{2}\cos(-p)t$, $\eta_2=\frac{a-b}{2}\sin(-p)t$.

The principles already enumerated enable us to combine two elliptical vibrations into a single one. Thus, if the component vibrations be represented by $\xi_1=a_1\cos(pt+\epsilon_1)$, $\eta_1=b_1\cos(pt+\epsilon_1-\Delta_1)$ and $\xi_2=a_2\cos(pt+\epsilon_2)$, $\eta_2=b_2\cos(pt+\epsilon_2-\Delta_2)$, the resultant vibration is $\xi=a\cos(pt+\epsilon)$, $\eta=b\cos(pt+\epsilon-\Delta)$, where

$$a^2=a_1^2+a_2^2+2a_1a_2\cos(\epsilon_1-\epsilon_2);$$

$$b^2=b_1^2+b_2^2+2b_1b_2\cos(\epsilon_1-\epsilon_2-\Delta_1+\Delta_2);$$

$$\tan\epsilon=\frac{a_1\sin\epsilon_1+a_2\sin\epsilon_2}{a_1\cos\epsilon_1+a_2\cos\epsilon_2};$$

$$\tan(\epsilon-\Delta)=\frac{b_1\sin(\epsilon_1-\Delta_1)+b_2\sin(\epsilon_2-\Delta_2)}{b_1\cos(\epsilon_1-\Delta_1)+b_2\cos(\epsilon_2-\Delta_2)}.$$

If the component vibrations are similar in form and sense, and have their corresponding axes parallel we have $\Delta_1=\Delta_2$ and $\frac{b_1}{a_1}=\frac{b_2}{a_2}$,

and in this case $\Delta = \Delta_1 = \Delta_2$ and $\frac{b}{a} = \frac{b_1}{a_1} = \frac{b_2}{a_2}$, so that the resultant vibration is similar in form and sense to the components.

Denoting the amplitudes of the component and resultant vibrations by r_1, r_2 and r we have

$$r^2 = r_1^2 + r_2^2 + 2\left[a_1 a_2 \cos(\epsilon_1 - \epsilon_2) + b_1 b_2 \cos(\epsilon_1 - \epsilon_2 - \Delta_1 + \Delta_2)\right].$$

The condition that the vibrations should be conjugate is $r^2 = r_1^2 + r_2^2$, i.e. $a_1 a_2 \cos(\epsilon_1 - \epsilon_2) + b_1 b_2 \cos(\epsilon_1 - \epsilon_2 - \Delta_1 + \Delta_2) = 0$. In order that this condition may be satisfied for all values of $\epsilon_1 - \epsilon_2$ we must have

$$a_1 a_2 + b_1 b_2 \cos(\Delta_1 - \Delta_2) = 0 \text{ and } b_1 b_2 \sin(\Delta_1 - \Delta_2) = 0,$$

whence, excluding the case of rectilinear vibrations already discussed, we must have $\sin(\Delta_1 - \Delta_2) = 0$ and $a_1 a_2 \pm b_1 b_2 = 0$, the upper sign being taken if $\Delta_1 - \Delta_2$ is an even multiple of π and the lower if it is an odd multiple. Putting $b_1/a_1 = \kappa$ we get $a_2 = \mp \kappa b_2$, and the component vibrations are represented by

$$\xi_1 = a_1 \cos(pt + \epsilon_1), \qquad \eta_1 = \kappa a_1 \cos(pt + \epsilon_1 - \Delta_1),$$

and $$\xi_2 = \mp \kappa b_2 \cos(pt + \epsilon_2), \quad \eta_2 = \pm b_2 \cos(pt + \epsilon_2 - \Delta_1).$$

From this it appears that the two ellipses are similar, have the same axes, the major axis of one being parallel to the minor axis of the other, and that the motion in one is clockwise and in the other counter-clockwise.

We shall conclude this chapter by a brief discussion of some of the most important features of natural or white light. The first point to notice is that this light is not homogeneous, but is proved by Newton's experiments with prisms to consist of various differently coloured beams superposed. The displacement must therefore be represented by the addition of a large number of terms corresponding to displacements in waves of different lengths. Here, however, the question naturally arises—what is the character of the vibration in each of the component waves, has it a definite orbit such as an ellipse? The answer forced upon us by the experimental evidence is that we must regard the light as polarised elliptically (to take the most general case possible) for an interval of time which is long compared with the period of a vibration, but very short compared with the

time required to make any impression on the retina or on a photographic plate. The form and orientation of the elliptic orbit is constantly changing so that in an appreciable interval of time the mean effect is that of a displacement that is perfectly symmetrical about the direction of propagation.

As regards polarisation, the distinguishing features of a stream of natural light revealed by experiment are:—(1) it can be resolved into two streams rectilinearly polarised at right angles to one another, (2) the intensities of these streams are independent of the azimuths of the planes of polarisation, and (3) if the streams be superposed once more the resultant is the same whatever be their relative retardation. These facts enable us to determine the conditions that must be satisfied by the displacements in a train of waves representing natural light. Thus suppose we are considering light propagated along the axis of z. It will appear later that the vibrations are always in the wave front so that the component displacements may be taken parallel to the axes of x and y. If the elliptic orbit in the nth constituent of the stream be such that its semi-axes are $a_n \cos \alpha_n$ and $a_n \sin \alpha_n$, and that its major axis makes an angle β_n with the axis of x, then the displacements parallel to the principal axes of the ellipse may be represented by

$$a_n \cos \alpha_n e^{i\theta_n} = A_n e^{i\theta_n} \text{ and } a_n \sin \alpha_n e^{i\left(\theta_n - \frac{\pi}{2}\right)}$$
$$= - i a_n \sin \alpha_n e^{i\theta_n} = - i B_n e^{i\theta_n}.$$

Resolving along the axes of x and y and summing the results for all the constituents we get as the components of the resultant displacement

$$\xi = \Sigma (A_n \cos \beta_n + i B_n \sin \beta_n) e^{i\theta_n},$$

and

$$\eta = \Sigma (A_n \sin \beta_n - i B_n \cos \beta_n) e^{i\theta_n}.$$

If now the component parallel to the axis of y receive a retardation Δ and the light be then resolved into two components in directions making angles ϕ and $90° + \phi$ with the axis of x, the displacement in the direction ϕ is represented by

$$\Sigma [A_n (\cos \beta_n \cos \phi + \sin \beta_n \sin \phi \, e^{-i\Delta})$$
$$+ i B_n (\sin \beta_n \cos \phi - \cos \beta_n \sin \phi \, e^{-i\Delta})] e^{i\theta_n}.$$

The intensity I is thus given by

$$\begin{aligned} I &= \Sigma\,(A_n^2\cos^2\beta_n + B_n^2\sin^2\beta_n)\cos^2\phi \\ &\quad + \Sigma\,(A_n^2\sin^2\beta_n + B_n^2\cos^2\beta_n)\sin^2\phi \\ &\quad + \sin\phi\cos\phi\,[2\Sigma\,(A_n^2 - B_n^2)\sin\beta_n\cos\beta_n\cos\Delta - 2\Sigma A_nB_n\sin\Delta] \\ &= (A+B)\cos^2\phi + (A-B)\sin^2\phi + \sin 2\phi\,(C\cos\Delta - D\sin\Delta) \end{aligned}$$

where
$$\begin{aligned} 2A &= \Sigma\,(A_n^2 + B_n^2) = \Sigma a_n^2\,; \\ 2B &= \Sigma\,(A_n^2 - B_n^2)\cos 2\beta_n = \Sigma a_n^2\cos 2\alpha_n\cos 2\beta_n\,; \\ 2C &= \Sigma\,(A_n^2 - B_n^2)\sin 2\beta_n = \Sigma a_n^2\cos 2\alpha_n\sin 2\beta_n\,; \\ 2D &= 2\Sigma A_nB_n = \Sigma a_n^2\sin 2\alpha_n. \end{aligned}$$

The condition that the stream thus represented should be equivalent to natural light is that I should be independent of ϕ for all values of Δ. This requires $B = C = D = 0$. A case of special interest is that in which we have two polarised streams. The elliptic orbit is constant in form and orientation for all the constituents of each stream, so that α_n and β_n are equal to α' and β' for one stream and to α'' and β'' for the other. We thus have

$$\begin{aligned} 2B &= \cos 2\alpha'\cos 2\beta'\,.\,\Sigma a_n'^2 + \cos 2\alpha''\cos 2\beta''\Sigma a_n''^2, \\ 2C &= \cos 2\alpha'\sin 2\beta'\Sigma a_n'^2 + \cos 2\alpha''\sin 2\beta''\Sigma a_n''^2, \end{aligned}$$

and
$$2D = \sin 2\alpha'\Sigma a_n'^2 + \sin 2\alpha''\Sigma a_n''^2.$$

The intensities, as we shall see, are proportional to $\Sigma a_n'^2$ and $\Sigma a_n''^2$, so that if $\kappa^2:1$ be the ratio of the intensities the conditions $B = C = D = 0$ give

$$\begin{aligned} \cos 2\alpha'\cos 2\beta' + \kappa^2\cos 2\alpha''\cos 2\beta'' &= 0, \\ \cos 2\alpha'\sin 2\beta' + \kappa^2\cos 2\alpha''\sin 2\beta'' &= 0, \end{aligned}$$

and
$$\sin 2\alpha' + \kappa^2\sin 2\alpha'' = 0.$$

Hence we must have $\kappa^2 = 1$, so that the two streams are of equal intensity. Putting $\kappa^2 = 1$ and confining ourselves to values of α and β which represent distinct ellipses we get $\alpha'' = -\alpha'$ and $\beta'' = \beta' + 90°$. Thus the two ellipses are similar, are described in opposite senses and have their major axes at right angles. The two vibrations are therefore conjugates. The fact that natural light may be represented by two such streams of equal intensity will be found of frequent use in the development of the theory.

CHAPTER III.

PROPAGATION OF LIGHT IN TRANSPARENT ISOTROPIC MEDIA. REFLECTION AND REFRACTION.

In the discussion of optical problems two vectors will appear throughout. These are the displacement at any point of the medium and the curl of the displacement. At any point (x, y, z) the displacement may be represented by its components (ξ, η, ζ), and the curl by its components (f, g, h), the relation between the two vectors being a geometrical one indicated by the equations

$$f = \frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z}, \quad g = \frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}, \quad h = \frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}.$$

Before we can apply the dynamical Principle of Action to the discussion of the propagation of a disturbance in the ether, it will be necessary to obtain expressions for the kinetic energy T and the potential energy W in terms of the vectors (ξ, η, ζ) and (f, g, h). Taking the density of the ether as the unit density we have

$$T = \tfrac{1}{2} \int (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)\, d\tau,$$

where $d\tau$ is an element of volume. The form of W will depend on the assumptions made as to the nature of the elasticity of the etherial medium. We shall suppose that the ether offers resistance to any rotation, but not to any translation, of its elements*. The work function for such a medium must, if the medium be isotropic, be given by an expression of the form

$$W = \tfrac{1}{2} c^2 \int (f^2 + g^2 + h^2)\, d\tau.$$

Applying the Principle of Action to any portion of the ether bounded by a surface S we have to make

$$\delta \int (T - W)\, dt = 0.$$

* This type of ether is analogous to the Elastic Solid Theory of Green and MacCullagh. For a complete discussion see Stokes: *Collected Works*, IV. p. 177.

Making use of the ordinary rules of variation we get

$$\delta \int (T - W)\, dt = \iint (\dot{\xi}\delta\dot{\xi} - c^2 f \delta f)\, dt\, d\tau + \mathfrak{Z}$$

two similar terms involving η and ζ. Also we have

$$\int \dot{\xi}\delta\dot{\xi}\, dt = \dot{\xi}\delta\xi - \int \ddot{\xi}\delta\xi\, dt$$

while the term containing $\delta\xi$ in the variation of δW is

$$c^2 \int \left(g \frac{\partial \delta\xi}{\partial z} - h \frac{\partial \delta\xi}{\partial y} \right) d\tau = c^2 \int (ng - mh)\, \delta\xi\, dS$$
$$- c^2 \int \left(\frac{\partial g}{\partial z} - \frac{\partial h}{\partial y} \right) \delta\xi\, d\tau,$$

where (l, m, n) are the direction cosines of the outward normal to the bounding surface S.

Picking out the coefficients of $\delta\xi$, $\delta\eta$, and $\delta\zeta$ in the variation, we obtain the three dynamical equations

$$\ddot{\xi} = c^2 \left(\frac{\partial g}{\partial z} - \frac{\partial h}{\partial y} \right), \quad \ddot{\eta} = c^2 \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z} \right), \quad \ddot{\zeta} = c^2 \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right),$$

while the surface conditions require $c^2(ng - mh)$, $c^2(lh - nf)$, and $c^2(mf - lg)$ to be continuous. Of course the displacement (ξ, η, ζ) must also be continuous if there is to be no rupture of the medium.

In the domain of optics we are interested only in periodic functions of the time, so that we may put everything proportional to e^{ipt}, where $2\pi/p$ is the period. We then have $\ddot{\xi} = -p^2\xi$ and so for $\ddot{\eta}$ and $\ddot{\zeta}$. Thus the dynamical equations may be written in the form

$$\xi = \frac{c^2}{p^2} \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right), \quad \eta = \frac{c^2}{p^2} \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right), \quad \zeta = \frac{c^2}{p^2} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right).$$

This shows that (ξ, η, ζ) may be regarded as the curl of a vector

$$\frac{c^2}{p^2}(f, g, h),$$

and that
$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0.$$

Thus, whether the medium be compressible or not, there is no compression involved in the motion.

Returning to the dynamical equations in their more general form we have

$$\ddot{\xi} = c^2\left(\frac{\partial g}{\partial z} - \frac{\partial h}{\partial y}\right) = c^2\left[\frac{\partial}{\partial z}\left(\frac{\partial \xi}{\partial z} - \frac{\partial \zeta}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right)\right]$$

$$= c^2\left[\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2}\right] - c^2\frac{\partial}{\partial x}\left[\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right] = c^2\nabla^2\xi,$$

with similar equations for η and ζ. The solution of these equations in various systems of coordinates is a familiar problem to the student of mathematics, but for our present purposes we are concerned only with some simple types of solution. If we are dealing with plane waves moving in any direction we may take this direction as that of the axis of z, and this without loss of generality since the medium is isotropic. Under such circumstances we have to solve equations of the type

$$\ddot{\xi} = c^2\frac{\partial^2 \xi}{\partial z^2},$$

the most general solution of which is

$$\xi = F_1(ct - z) + F_2(ct + z),$$

where F_1 and F_2 are arbitrary functions. This represents waves of arbitrary form moving with velocity c, in the positive and negative directions of the axis of z. Hence the quantity c measures the velocity of light in free ether.

As was pointed out in the last chapter we are specially concerned with wave forms represented by a sine or cosine curve. For such forms we have, for progressive waves moving in the positive direction of the axis of z,

$$(\xi, \eta, \zeta) = (L, M, N)\, Ae^{ip(t - z/c)}.$$

If LMN be real, the light is plane polarised, and LMN are the direction cosines of the direction of displacement. If, on the other hand, LMN be complex, the polarisation is elliptical, and the real parts of LMN are the direction cosines of one axis and the imaginary parts those of the other axis of the elliptic orbit. Since

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0,$$

we have $N = 0$, so that the displacement is in the wave front.

The axes of x and y have not yet been fixed, and by taking them parallel to the principal axes of the elliptic orbit we may put $L=1$ and $M=0$, we then have $\eta=\zeta=0$ and $\xi=Ae^{ip(t-z/c)}$, from which we find

$$f=0,\quad g=-\frac{ip}{c}Ae^{ip(t-z/c)},\quad h=0.$$

Hence the curl of the displacement is also in the wave front, and its amplitude is proportional to that of the displacement, the magnitude of the former being p/c times that of the latter.

The kinetic energy per unit volume is

$$T=\tfrac{1}{2}(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)=\tfrac{1}{2}\dot{\xi}^2=-\frac{p^2A^2}{2}e^{2ip(t-z/c)}.$$

Similarly the potential energy per unit volume is

$$W=\tfrac{1}{2}c^2(f^2+g^2+h^2)=-\frac{p^2A^2}{2}e^{2ip(t-z/c)}.$$

Hence the potential and kinetic energies per unit volume are the same. The total energy E in a tube of unit cross section extending along the axis of z from the origin to $z=z_1$, is

$$E=T+W=2T=-p^2A^2\int_0^{z_1}e^{2ip(t-z/c)}\,dz.$$

From this we obtain

$$\frac{dE}{dt}=-2ip^3A^2\int_0^{z_1}e^{2ip(t-z/c)}\,dz=-p^2A^2\left[e^{2ip(t-z/c)}\right]_0^{z_1}.$$

Hence the energy may be regarded as flowing across $z=z_1$ at the rate $p^2A^2e^{2ip(t-z_1/c)}$, which is the same quantity as the total energy per unit volume in the neighbourhood of z_1. Thus in comparing the intensities of two beams we shall get the same ratio whether we measure the intensity by the mean potential energy, the mean kinetic energy, the mean energy, or the rate at which energy is flowing across a surface.

Since the velocity is a constant (c), we see that the wave surface is a sphere and the radius vector drawn from any origin to represent a ray is normal to the sphere. Hence the ray and the wave normal coincide. As there is perfect symmetry round the origin, the energy must flow along the radius vector and so along the ray. Thus the two definitions of a ray suggested in Chap. II. are identical.

The formulae just obtained for the energy have reference to progressive plane waves moving along some fixed direction. These formulae must be modified if we have simultaneously two trains of waves moving in opposite directions. A specially important case of this occurs when there are two waves of the same amplitude and frequency. If θ be the difference of phase between the waves when $z=0$, the displacement ξ is given by the equation

$$\xi = A\cos p(t-z/c) + A\cos[p(t+z/c)+\theta]$$
$$= 2A\cos(pt+\theta/2)\cos(pz/c+\theta/2).$$

Under such circumstances the phase is constant everywhere, but the amplitude is

$$\pm 2A\cos(pz/c+\theta/2),$$

which is a periodic function of z. Waves of this kind are called *stationary waves*, and the points where the amplitude is a maximum are *loops*, and those where it is zero are *nodes* in the waves. The loops occur when

$$pz/c+\theta/2 = n\pi,$$

i.e. when
$$z = n\lambda/2 - \lambda\theta/4\pi,$$

where λ is the wave length, and the nodes occur when

$$z = (2n+1)\lambda/4 - \lambda\theta/4\pi.$$

Thus the distance between successive loops or successive nodes is $\lambda/2$. Taking

$$\xi = A\cos p(t-z/c) + A\cos\{p(t+z/c)+\theta\};\ \eta=0;\ \text{and}\ \zeta=0,$$

we get

$$f=0,\ h=0,\ \text{and}\ g = pA/c\,.\,[\sin p(t-z/c) - \sin\{p(t+z/c)+\theta\}].$$

Hence

$$T = \dot{\xi}^2/2 = p^2A^2/2\,.\,[\sin p(t-z/c) + \sin\{p(t+z/c)+\theta\}]^2,$$

and

$$W = c^2g^2/2 = p^2A^2/2\,.\,[\sin p(t-z/c) - \sin\{p(t+z/c)+\theta\}]^2.$$

Thus since

$$\int_0^{2\pi/p} \sin^2 p(t-z/c)\,dt = \pi/p = \int_0^{2\pi/p} \sin^2\{p(t-z/c)+\theta\}\,dt,$$

and

$$\int_0^{2\pi/p} \sin p(t-z/c)\sin\{p(t+z/c)+\theta\}\,dt = \pi/p\,.\cos(2pz/c+\theta),$$

we have $$\bar{T} = 2\pi p A^2 \cos^2(pz/c + \theta/2),$$

and $$\overline{W} = 2\pi p A^2 \sin^2(pz/c + \theta/2),$$

where $\bar{T}$ and $\overline{W}$ are the mean values of T and W throughout a period. It is therefore no longer true that the mean value of the kinetic energy is equal to that of the potential energy. $\bar{T}$ and $\overline{W}$ are both periodic functions of z, $\bar{T}$ vanishing at the nodes and $\overline{W}$ at the loops of the stationary waves.

So far in this chapter we have been considering the propagation of light in free ether. When we come to discuss the corresponding problems for material media, and to deal with reflection and refraction at the surface of a transparent isotropic body, we are faced with the difficulties connected with the relations between ether and matter. However, it will not be necessary to delay our progress until these difficulties are overcome. A large amount of experimental evidence goes to show that matter is not a continuum, but is made up of discrete particles or atoms; but even if we were satisfied with our knowledge of the nature of these atoms, we should not be concerned at this stage with the action of the individual atoms on a wave of light. We are interested only in their average effect, the influence of all the atoms in a finite portion of matter. We assume as a working hypothesis that the presence of matter affects the ether in such a way that ether and matter may be replaced ideally by a continuum whose rotational elasticity is different from that of the free ether, but whose density is the same. The elastic constant of this medium obtained by mentally smoothing out the atoms and spreading their influence uniformly throughout the body is no longer c as for free ether, but some other constant c/μ. With this slight change all the results previously obtained for the propagation of light in the ether apply to its propagation in any transparent isotropic body. In particular, the dynamical equations become

$$\ddot{\xi} = \frac{c^2}{\mu^2}\left(\frac{\partial g}{\partial z} - \frac{\partial h}{\partial y}\right), \quad \ddot{\eta} = \frac{c^2}{\mu^2}\left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right), \quad \ddot{\zeta} = \frac{c^2}{\mu^2}\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right);$$

the boundary conditions require the continuity of

$$\xi, \; \eta, \; \zeta, \; \frac{c^2}{\mu^2}(ng - mh), \; \frac{c^2}{\mu^2}(lh - nf), \; \frac{c^2}{\mu^2}(mf - lg);$$

and the velocity of propagation is c/μ. The quantity μ will be identified with the refractive index of the medium.

In applying these formulae to the problem of reflection and refraction we shall confine ourselves in this chapter to the ideal case of an abrupt transition from one medium to the other. The sequel will prove that in many cases actually presented experimentally the transition is not of this abrupt character, and the formulae will require modification before they can be applied to such cases. The nature of this modification will be considered in the next chapter.

In dealing with reflection and refraction it is convenient to begin with a discussion of plane polarised light and to consider separately the cases where the displacements are perpendicular and parallel to the plane of incidence. The results for any other case can be deduced from these by compounding vibrations on the principles explained in the last chapter. We shall take the surface of separation to be $x=0$, the plane of xy to be the plane of incidence, and then all the vectors are independent of z.

Let the displacement (ξ, η, ζ) be perpendicular to the plane of incidence and consequently the curl (f, g, h) parallel to this

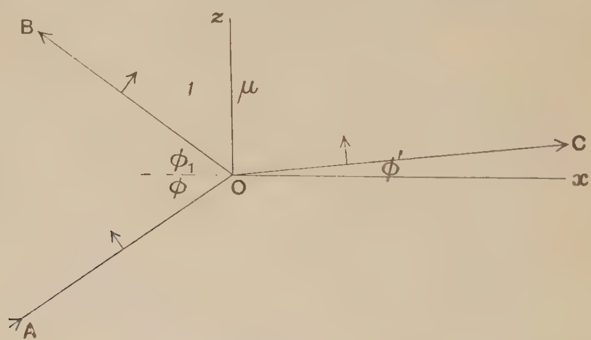

Fig. 6.

plane. Let AO, OB and OC represent the incident, reflected and refracted rays, ϕ, ϕ_1 and ϕ' the angles of incidence, reflection and refraction. Let the arrows indicate the positive direction of the *curl* (f, g, h), these being chosen so as all to be parallel and in the same sense in the case of normal incidence. Taking the refractive index of the first medium to be unity, let μ be that of the second.

We have $\xi = 0 = \eta$, and in the first medium

$$\zeta = e^{ip[t - (x\cos\phi + y\sin\phi)/c]} - re^{ip[t - (x\cos\overline{\pi - \phi_1} + y\sin\overline{\pi - \phi_1})/c]}$$
$$= e^{ip[t - (x\cos\phi + y\sin\phi)/c]} - re^{ip[t + (x\cos\phi_1 - y\sin\phi_1)/c]},$$

the first term corresponding to the incident and the second to the reflected wave. In the second medium we have

$$\zeta = se^{ip[t - (x\cos\phi' + y\sin\phi')\mu/c]},$$

which represents the refracted wave. In writing down these formulae we have made use of the theorems already proved, that the ray coincides with the wave normal and that the velocity of propagation of the wave is c/μ. The boundary conditions require ζ to be continuous at the interface $x = 0$ for all values of y and t. Hence we must have $\sin\phi = \sin\phi_1 = \mu\sin\phi'$. These are the fundamental laws of reflection and refraction, amply verified by experiment, and familiar to everyone acquainted with the elements of the science of optics. The law connecting the angles of incidence and refraction is sometimes ascribed to Snell, and sometimes to Descartes, who first stated it in the form here presented.

The dynamical principles already explained give us much more than Snell's law determining the directions of the reflected and refracted rays. They enable us to determine the amplitudes of the displacements and their curls in each of the waves, and thus to estimate their relative intensities. We have seen that for progressive waves, such as those now under discussion, it is immaterial whether we measure the intensity by the mean kinetic energy, the mean potential energy, or the rate at which energy is flowing across a surface. The kinetic energy per unit volume is $\frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) = \frac{1}{2}\dot{\zeta}^2$, in this case and this is proportional to ζ^2. Hence the intensities of the incident, reflected and refracted rays are in the ratios $1 : r^2 : s^2$. It may be noted that while the amplitudes of the displacements are in the ratios $1 : r : s$, those of the curls are in the ratios $1 : r : \mu s$. These various ratios are determined by means of the boundary conditions which require the continuity of ζ and of g/μ^2, i.e. of ζ and of $\dfrac{1}{\mu^2}\dfrac{\partial\zeta}{\partial x}$. Whence we get $1 - r = s$, and $\cos\phi\,(1 + r) = \dfrac{s}{\mu}\cos\phi'$, so that

$$r = \frac{\cos\phi' - \mu\cos\phi}{\cos\phi' + \mu\cos\phi} = -\frac{\tan(\phi - \phi')}{\tan(\phi + \phi')},$$

since $\sin\phi = \mu \sin\phi'$; and

$$s = \frac{2\mu\cos\phi}{\cos\phi' + \mu\cos\phi} = \frac{\sin 2\phi}{\sin(\phi+\phi')\cos(\phi-\phi')},$$

$$s_1 = \frac{s}{\mu} = \frac{2\sin\phi'\cos\phi}{\sin(\phi+\phi')\cos(\phi-\phi')}.$$

Before discussing these formulae we shall deal similarly with the other case in which the displacement (ξ, η, ζ) is parallel to the plane of incidence, and its curl (f, g, h) perpendicular to this plane. For this purpose it is convenient to introduce a new vector (ξ', η', ζ') of which (ξ, η, ζ) is the curl, the possibility of this having been pointed out before. We then have

$$\xi' = 0 = \eta'; \quad \xi = \frac{\partial\zeta'}{\partial y}, \; \eta = -\frac{\partial\zeta'}{\partial x}, \; \zeta = 0; \quad f = 0 = g; \quad h = -\nabla^2\zeta'.$$

In the first medium we may take

$$\zeta' = e^{ip[t-(x\cos\phi + y\sin\phi)/c]} + r' e^{ip[t+(x\cos\phi - y\sin\phi)/c]},$$

the first term representing the incident and the second the reflected wave. In the second medium we have similarly

$$\zeta' = \frac{s'}{\mu} e^{ip[t-(x\cos\phi' + y\sin\phi')\mu/c]}.$$

The continuity of the displacement at all points of $x=0$, leads as before to Snell's law $\sin\phi = \mu\sin\phi'$. The amplitudes of the displacements in the different waves are in the ratios $1 : r' : s'$, and those of the curls in the ratios $1 : r' : \mu s'$. The ratios of the intensities are $1 : r'^2 : s'^2$. As before these ratios are determined from the boundary conditions which require the continuity of ξ, η, and h/μ^2. From these we get

$$1 + r' = \frac{s'}{\mu} \text{ and } \cos\phi(1-r') = s'\cos\phi',$$

so that $$r' = \frac{\cos\phi - \mu\cos\phi'}{\cos\phi + \mu\cos\phi'} = -\frac{\sin(\phi-\phi')}{\sin(\phi+\phi')}$$

and $$s' = \frac{\sin 2\phi}{\sin(\phi+\phi')}; \quad s_1' = \frac{s'}{\mu} = \frac{2\sin\phi'\cos\phi}{\sin(\phi+\phi')}.$$

These formulae giving the intensities in the different waves are substantially the same as those first obtained by Fresnel, and they will be referred to as Fresnel's formulae*. The formulae for r and r' are the most interesting from the point of view of the

* Fresnel: *Œuvres* I. No. XXX. p. 774.

experimenter. The march of these functions is represented in Fig. 7, which gives the graphs of r^2 and r'^2 for ordinary glass

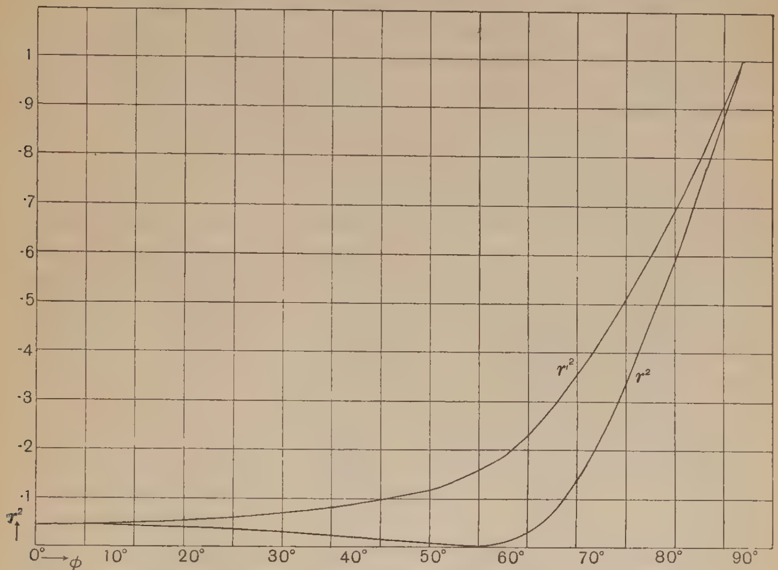

Fig. 7.

whose refractive index is $\mu = 1{\cdot}52$. From the formulae, or the graph, it appears that r'^2 and r^2 are equal at normal incidence (as they must be since there is then no distinction between directions parallel and perpendicular to the plane of incidence) and also at grazing incidence where their value is 1. As the angle of incidence increases r'^2 constantly increases, slowly at first, but afterwards somewhat rapidly. On the other hand r^2 begins by diminishing and continues in this course until it vanishes, when $\phi + \phi' = 90°$, i.e. when $\tan \phi = \mu$. The angle at which r vanishes is called the polarising angle for reasons that will appear presently. After this angle is passed r^2 begins to increase and continues to do so rapidly until it reaches the value unity at grazing incidence.

Let us consider a beam of natural light incident on any reflecting surface of a transparent isotropic body. We have seen that the light may be replaced analytically by two plane polarised streams of equal intensity, the planes of polarisation being at right angles. Thus we have two waves, the displacements in one being

parallel to the plane of incidence and those in the other at right angles to this plane. The amplitudes of these displacements are equal. After reflection, however, the amplitudes will no longer be equal. Let ϵ be the ratio of the amplitude of the displacement perpendicular to the plane of incidence to that of the displacement parallel thereto. Then by Fresnel's formulae we have

$$\epsilon = \pm r/r' = \pm \cos(\phi + \phi')/\cos(\phi - \phi'),$$

the sign being chosen so as to make ϵ positive, for ϵ being the ratio of two amplitudes cannot be negative. When $\phi + \phi' = 90^\circ$, i.e. when $\tan\phi = \mu$, both r and ϵ vanish. Thus the displacements perpendicular to the plane of incidence are wanting from the reflected beam, in other words, the reflected light is polarised in the plane of incidence. The fact that natural light can thus be polarised by reflection was first observed by Malus, and a few years later Brewster* deduced from his experiments that the polarising angle is given by the formula $\tan\phi = \mu$. More modern experiments indicate a very slight departure from Brewster's law, but they show that the differences between observation and calculation are extremely small when the reflecting surfaces are thoroughly clean and freshly polished. An explanation of the slight discrepancy will appear in the next chapter, when the influence of the layer of transition from one medium to the other is considered. The following table gives the polarising angles of various substances calculated from Brewster's law and compared with experiment.

Theory	53°7′	53°18′	55°33′	55°37′	58°13′	58°36′	59°41′	60°30′	63°33′	67°7′	67°32′	67°40′
Experiment	53°7′	53°18′	55°33′	55°37′	58°12′	58°36′	59°44′	60°30′	63°34′	67°6′	67°26′	67°30′

The quantity denoted above by ϵ can be deduced from experiment and the results compared with those derived from theory. This comparison is made for reflection from a diamond in the following table and in the accompanying Fig. 8 which represents the results graphically.

In this figure, as in all that follow, the curve corresponds to the theory and the crosses to the experiments. The agreement is as close as could be desired except in the neighbourhood of the

* *Phil. Trans.* p. 123. 1815.

ϕ......	60°	61°	62°	63°	64°	65°
ϵ_2^2 (Theory)	·0421	·0324	·0234	·0166	·0104	·0056
ϵ^2 (Experiment)	·0420	·0312	·0213	·0178	·0102	·0057
Difference	+ ·0001	+ ·0012	+ ·0021	− ·0012	+ ·0002	− ·0001

ϕ......	70°	71°	72°	73°	74°	75°
ϵ_2^2 (Theory)	·0049	·0103	·0177	·0275	·0399	·0552
ϵ^2 (Experiment)	·0054	·0106	·0184	·0296	·0469	·0576
Difference	− ·0005	− ·0003	− ·0007	− ·0021	− ·0070	− ·0024

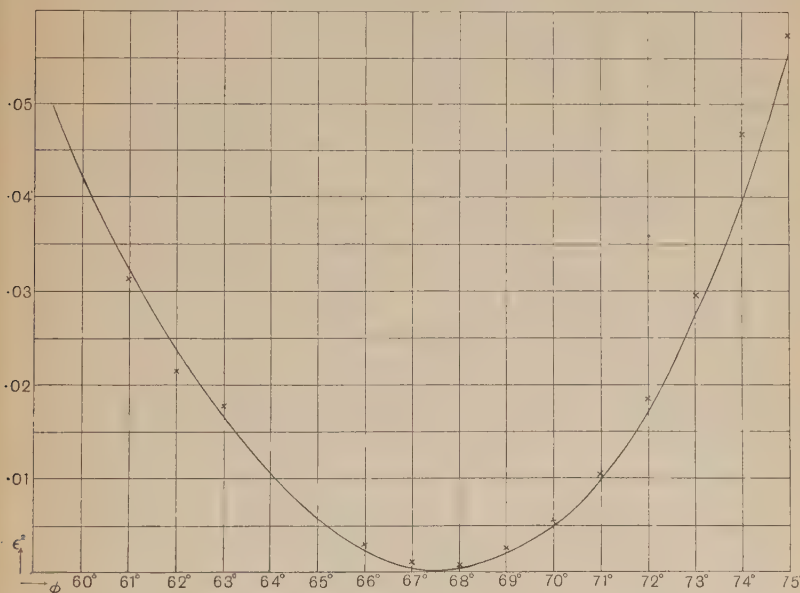

Fig. 8.

polarising angle. The explanation of the discrepancy in this region will be discussed in the next chapter.

Fresnel's formulae give us the amplitudes of the displacements and their curls for vibrations confined either to the plane of incidence or to the plane perpendicular to this. It is, however,

a simple matter to derive from them formulae for these amplitudes when the incident vibrations are in any given azimuth. Thus if the displacement in the incident wave be of amplitude A, and make an angle θ with the plane of incidence, we may resolve this vector into its components $A\cos\theta$ and $A\sin\theta$ respectively parallel and perpendicular to the plane of incidence. If then the displacement in the reflected wave be of amplitude A' and it make an angle θ' with the plane of incidence, Fresnel's formulae give us

$$A'\cos\theta' = r'A\cos\theta = -A\cos\theta \, . \frac{\sin(\phi-\phi')}{\sin(\phi+\phi')} \quad \ldots\ldots(1),$$

and $$A'\sin\theta' = rA\sin\theta = -A\sin\theta \, . \frac{\tan(\phi-\phi')}{\tan(\phi+\phi')} \quad \ldots\ldots(2),$$

which determine A' and θ'.

If we eliminate θ from (1) and (2) we get

$$\frac{A^2}{A'^2} = \frac{\cos^2\theta'}{r'^2} + \frac{\sin^2\theta}{r^2},$$

from which it follows that if we keep the amplitude (A) constant, but vary the azimuth (θ), the end of the line representing the displacement in the reflected wave traces out an ellipse. Also from (1) and (2) we obtain by division

$$\tan\theta' = \frac{\sec(\phi-\phi')}{\sec(\phi+\phi')}\tan\theta.$$

Differentiating this and making use of (1) and (2) we find

$$A'^2 d\theta' = rr' \, . A^2 d\theta,$$

so that the area swept out by a radius vector representing the displacement in the reflected wave is proportional to that swept out by the corresponding vector for the incident wave.

The relation $$\tan\theta' = \frac{\sec(\phi-\phi')}{\sec(\phi+\phi')}\tan\theta,$$

can be put to the test of comparison with experiment by measuring the azimuths of the planes of polarisation of the incident and reflected waves for different angles of incidence. The two following tables and figures exhibit the results of such a comparison, the experiments being those of Brewster* made in 1830, and the refractive index 1·4826.

* *Phil. Trans.* p. 74. 1830.

ϕ......	10°	20°	40°	50°	60°	70°	80°	86°
θ' (Theory)	43° 49′	40° 4′	23° 1′	9°	6° 16′	21° 3′	33° 46′	40° 36′
θ' (Experiment)	44°	39°	22° 37′	9°	6° 10′	22° 6′	33° 13′	40° 43′
Difference	− 0° 11′	+ 1° 4′	+ 0° 24′	0°	+ 0° 6′	− 1° 3′	+ 0° 33′	− 0° 7′

This table shows how θ' varies with the incidence, θ being constant and equal to 45° throughout. Fig. 9 below represents the same results graphically.

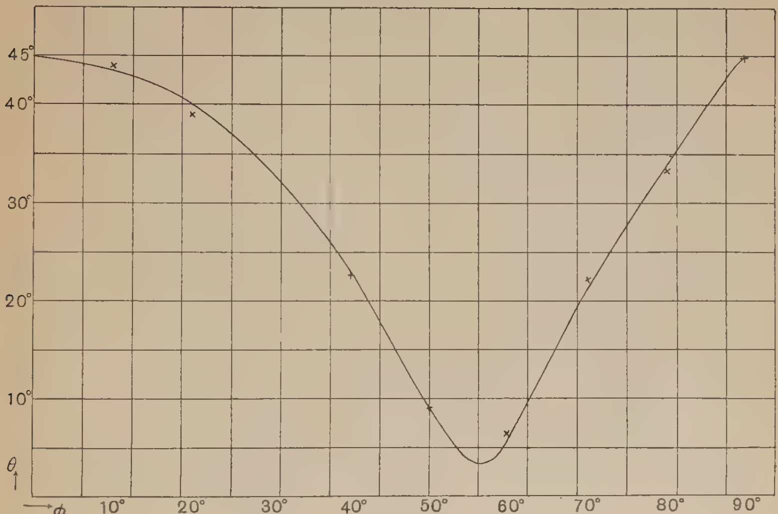

Fig. 9.

The next table shows how θ' varies with θ when the angle of incidence is constant and equal to 75° throughout. Fig. 10 exhibits the same results graphically.

θ......	10°	20°	35°	45°	55°	60°	70°	80°
θ' (Theory)	4° 29′	10° 16′	19° 12′	26° 27′	35° 23′	40° 45′	53° 49′	70° 29′
θ' (Experiment)	4° 54′	10°	20°	26° 20′	35° 30′	40°	53°	70°
Difference	− 0° 25′	+ 0° 16′	− 0° 48′	+ 0° 7′	− 0° 7′	+ 0° 45′	+ 0° 49′	+ 0° 29′

The experiments were not very refined and the differences between observation and calculation were nearly all within the

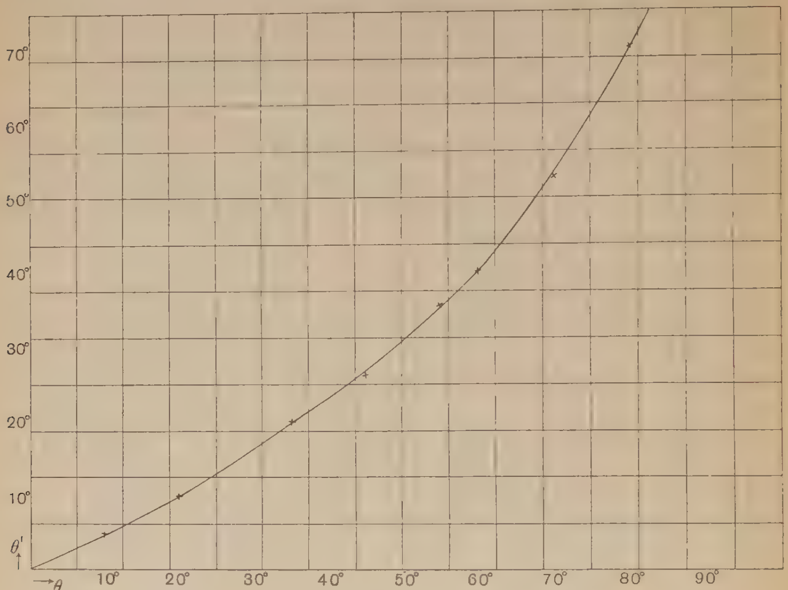

Fig. 10.

limits of the errors of experiment. We shall see in the next chapter that if the transition from one medium to the other is gradual and not abrupt, the reflected beam is elliptically polarised instead of being plane polarised. Under most circumstances, however, the minor axis of the ellipse is extremely small, and the azimuth of the major axis is very near that of the plane of polarisation calculated on the hypothesis of an abrupt transition.

The refracted wave may be dealt with on exactly similar lines. Thus if the amplitude of the displacement be A_1 and its azimuth θ_1 we have

$$A_1 \cos\theta_1 = s' A \cos\theta \quad \text{and} \quad A_1 \sin\theta_1 = s \,.\, A \sin\theta,$$

which determine A_1 and θ_1 when s and s' are known from Fresnel's formulae. The azimuth of the plane of polarisation of the refracted wave is given by the equation

$$\tan\theta_1 = \frac{s}{s'}\tan\theta = \sec(\phi - \phi')\tan\theta.$$

The two tables and figures that follow compare the results obtained from this formula with those found experimentally by Brewster.

ϕ......	10°	20°	30°	40°	45°	55°	65°	80°
θ_1 (Theory)	45° 6′	45° 25′	46°	46° 56′	47° 34′	48° 59′	52° 7′	59° 5′
θ_1 (Experiment)	45° 13′	45° 27′	45° 40′	46° 30′	46° 47′	48° 54′	51° 48′	58° 40′
Difference	−0° 7′	−0° 2′	+0° 20′	+0° 26′	+0° 47′	+0° 5′	+0° 19′	+0° 25′

This with the corresponding Fig. 11 below shows how θ_1 varies with the incidence when $\theta = 45°$.

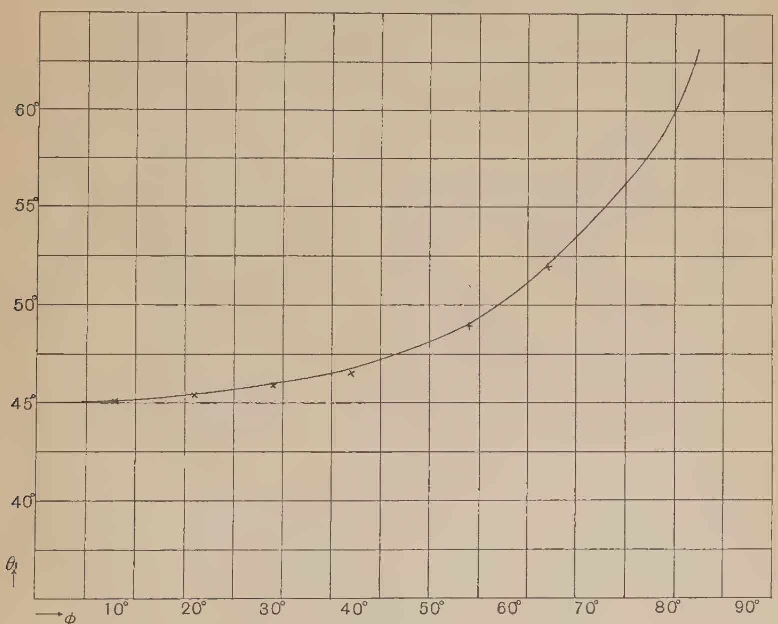

Fig. 11.

The next table shows how θ_1 varies with θ for a given angle of incidence, $\phi = 80°$. These results are shown graphically in Fig. 12.

The formulae

$$\tan \theta_1 = \tan \theta \,.\, \sec (\phi - \phi') = \tan \theta' \sec (\phi + \phi')$$

θ......	10°	20°	30°	40°	50°	55°	65°	80°
θ_1 (Theory)	16° 25′	31° 19′	43° 57′	54° 31′	63° 19′	67° 15′	74° 24′	83° 58′
θ_1 (Experiment)	17° 10′	32° 30′	44° 10′	54° 36′	63° 10′	66° 58′	74° 8′	83° 23′
Difference	− 0° 45′	− 1° 11′	− 0° 13′	− 0° 5′	+ 0° 9′	+ 0° 17′	+ 0° 16′	+ 0° 35′

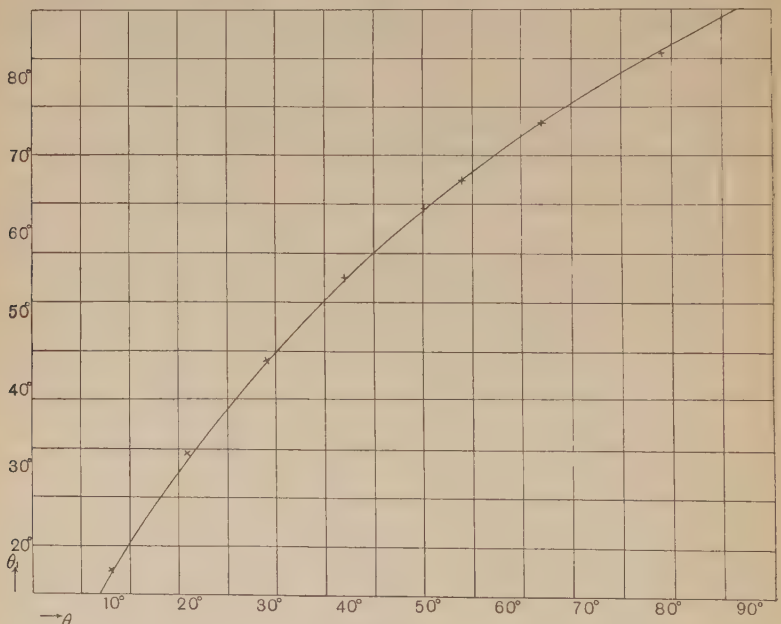

Fig. 12.

enable us to give a very simple geometrical construction for the direction of the displacement in the reflected and refracted waves when that in the incident wave is given. However, instead of deriving this construction from Fresnel's formulae we shall obtain it by another method which is more direct and will be of use when dealing with the corresponding problem of crystalline reflection and refraction. The displacement (ξ, η, ζ) is a vector whose square is proportional to the energy per unit volume, or to the rate at which energy is flowing across any surface. Hence, on the doctrine of energy which is involved in our fundamental hypotheses, the magnitude of this vector is not altered in going from one

medium to another. Moreover to avoid a rupture of the medium the components (ξ, η, ζ) must be continuous at an interface. Hence it follows that the refracted displacement is the resultant, in the mechanical sense, of the incident and reflected displacements. These three are therefore in one plane and are connected by the parallelogram law. If $Z = 0$ be the plane of incidence and O the origin, the three waves pass through the line OZ and are inclined

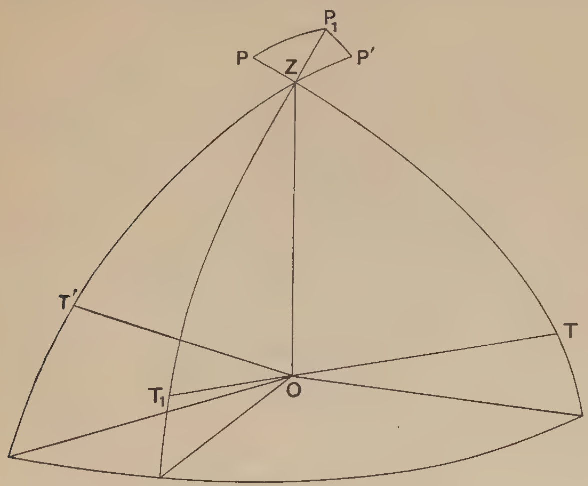

Fig. 13.

to one another at angles determined by Snell's law and the law of reflection. Thus if the incident wave be given the other waves are easily obtained, and we have now to investigate the directions of the displacements in these waves. Let OT be the direction of the incident displacement, OP the direction of its curl, so that TOP is a right angle, T and P being points on the unit sphere whose centre is O. Through OP draw a plane perpendicular to the incident wave to cut the refracted wave in OP_1, and through OP_1 draw a plane perpendicular to the reflected wave to cut it in OP'. Then OP' and OP_1 are the directions of the curls in the reflected and refracted waves and OT' and OT_1, at right angles to these lines, are the directions of the displacements. Since TP and TPP_1 are right angles, P_1T is a right angle. Similarly P_1T' is a right angle. Hence P_1 is the pole of TT_1T',

so that the plane of the three displacements is the polar plane of the curl of the refracted displacement.

The same general principles also enable us to prove, independently of Fresnel's formulae, the important theorem that at the polarising angle the angles of incidence and refraction are complementary. Let OT (Fig. 14) be the section of the interface by the

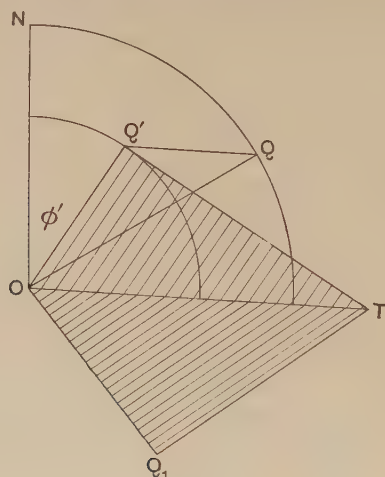

Fig. 14.

plane of incidence, ON the normal to the interface, OQ and OQ' the directions of the incident and refracted rays. If OQ_1 be parallel to the front of an incident wave, the direction of the wave front is altered so that after refraction it becomes TQ'. If TQ be parallel to OQ_1 then TQ and TQ' are, by Huyghens' construction, tangents to the circles whose radii are proportional to the velocities of propagation of the waves in the two media. Hence the volumes of disturbance in the two media are proportional to the areas of the triangles $OQ'T$ and OQ_1T, i.e. $OQ'T$ and OQT, since $OQT = OQ_1T$. We have seen that the refracted displacement is the resultant of the incident and reflected displacements. Thus at the polarising angle where there is no reflected displacement the incident and refracted displacements are equal in magnitude and direction. Hence the energy per unit volume is the same in the two waves, and therefore, since there is no change of energy, the areas OQT

and $OQ'T$ are equal and OQ' is parallel to OT. From this it easily follows that the angles ϕ and ϕ' are complementary. For $OQ'QT$ can be inscribed in a circle of which ON is the tangent at O. Hence

$$\phi' = Q'ON = \text{angle in alternate segment} = OQQ' = QOT = 90° - \phi.$$

Returning to the consideration of Fresnel's formulae we see that at normal incidence

$$r = r' = -\frac{\mu - 1}{\mu + 1},$$

so that these terms are negative. The displacements in the incident and reflected waves are represented by $e^{ip\,(t-x/c)}$ and $-re^{ip\,(t+x/c)}$. Hence as r is negative, the displacements in the two waves are in the same direction and there is no change of phase on reflection. As the angle of incidence increases r and r' take different values, but r' retains the same sign throughout. Thus for light polarised parallel to the plane of incidence there is no change of phase throughout. It is different, however, for light polarised at right angles to the plane of incidence, for r vanishes at the polarising angle and changes its sign there. Hence there is a sudden change of phase of half a wave length on passing through the polarising angle. If we are dealing with the curl of the displacement it must be noted that in Fig. 6 the directions were so chosen that the curls in the incident and reflected waves were taken positively in the same direction for normal incidence. Hence the negative value of r at normal incidence shows that the curls are in opposite directions in the two waves, or that there is a change of phase of half a wave length on reflection.

For purposes of comparison with experiment we are usually more interested in the difference of phase after reflection between the components parallel and perpendicular to the plane of incidence than with the actual change of phase for either of these components. We shall put $r = Re^{i\rho}$ and $r' = R'e^{i\rho'}$, so that

$$r/r' = R/R' \, . \, e^{i(\rho - \rho')} = \epsilon \, . \, e^{i\Delta},$$

where $R/R' = \epsilon$ and $\Delta = \rho - \rho'$. If the displacement in the incident wave be resolved into components of unit amplitudes parallel and perpendicular to the plane of incidence, then after reflection the component displacements are represented by

$\xi_1 = R' \cos(pt + \rho')$, and $\eta_1 = R \cos(pt + \rho)$. If then ρ be greater than ρ' the crest (or hollow) of the wave η_1 is reached sooner than the crest (or hollow) of the wave ξ_1. Thus ξ_1 lags behind η_1, the amount of the lag being measured by $\rho - \rho' = \Delta$. According to Fresnel's formulae Δ is zero for all angles of incidence less than the polarising angle. At this angle Δ changes suddenly to π or $-\pi$, so that there is an acceleration or retardation of phase of half a wave length. It is found, however, by experiment that the change of phase does not take place suddenly but enters by degrees. The change is very rapid in the neighbourhood of the polarising angle and very slow outside of that neighbourhood. At the polarising angle itself Δ is found to be very nearly $\pi/2$. Fig. 15 represents the results obtained by reflection from diamond,

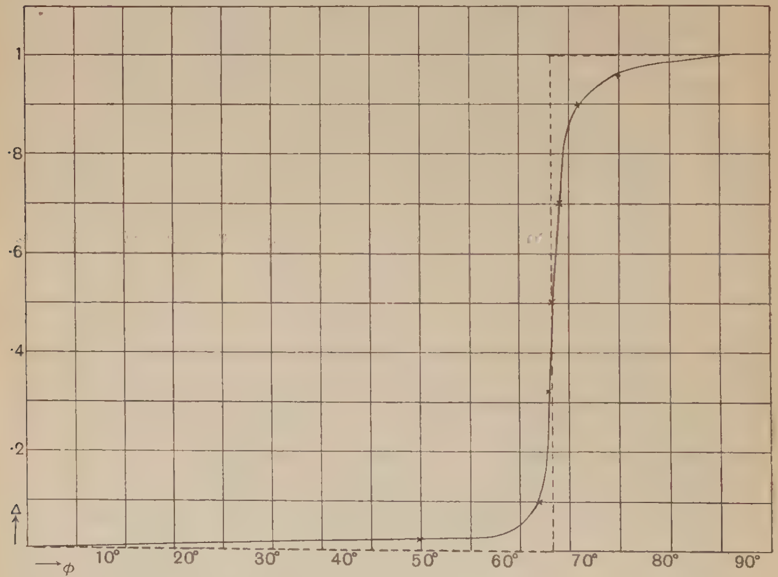

Fig. 15.

the dotted line corresponding to Fresnel's formulae and the crosses along the continuous curve to the experimental results. Δ increases from zero to π throughout the range. Under such circumstances the reflection is said to be negative, to distinguish this from what is called positive reflection, when Δ diminishes from 0 to $-\pi$.

The intensity of the light reflected from a transparent isotropic body is easily calculated from Fresnel's formulae. If natural light of unit intensity be employed, we can replace it by two beams whose intensity is $\frac{1}{2}$, polarised in planes parallel and perpendicular to the plane of incidence. The intensity of the reflected light is thus $(r^2+r'^2)/2$, which at normal incidence becomes $[(\mu-1)/(\mu+1)]^2$. In 1870 Rood* concluded from his experiments that "the reflecting power of glass conforms in the closest manner to the predictions of theory" or to Fresnel's formulae. However, in 1886, this conclusion was shown to be untenable by Lord Rayleigh†. The difficulties of measuring the intensity of the reflected beam directly are very considerable, and Rood had contented himself with measuring the transmitted light and deducing the amount that was reflected. Rayleigh showed that when this fact was considered the difference between theory and Rood's experimental results might amount to 7 % of the reflected light, a difference much too great to be regarded as insignificant. Rayleigh found from his own experiments that recently polished glass surfaces have a reflecting power differing not more than 1 % or 2 % from that calculated from Fresnel's formulae; but that after some months or years the reflection may fall off 10 % or more and that without any apparent tarnish. About the same time Conroy carried out a careful series of experiments on the same subject, his results confirming those of Lord Rayleigh. The following table gives Conroy's estimates of the percentage of light reflected from glass that had been polished a considerable time before the experiments were made. It also compares the results with the calculations from Fresnel's formulae.

ϕ......	0°	10°	20°	30°	40°	50°	60°	65°	70°
Reflecting power (Experiment)	·0378	·0378	·0377	·0392	·0437	·0553	·0854	·1116	·1604
(Theory)	·0419	·0419	·0421	·0434	·0477	·0598	·0916	·1231	·1737
Difference	·0041	·0041	·0044	·0042	·0040	·0045	·0062	·0115	·0133

* *Amer. Journ. Sci.* Vol. XLIX. 1870 (March). Vol. L. 1870 (July).

† *Scientific Papers*, Vol. II. p. 523. *Proc. Roy. Soc.* XLI.

These results are exhibited graphically in Fig. 16.

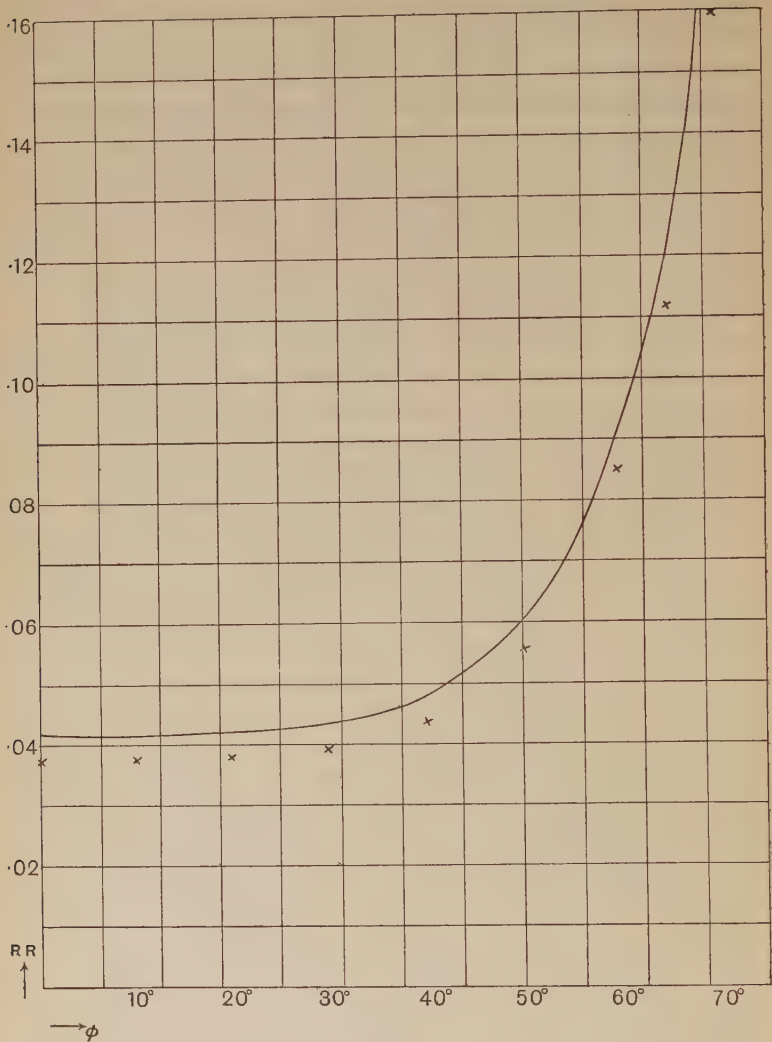

Fig. 16.

There can thus be no doubt of a decided departure from Fresnel's formulae in certain circumstances. The difference between observation and calculation is too great to be put down to experimental errors, and there is no evidence of such errors

being considerable, seeing that the results of experiment are fairly consistent. Nor can there be very much doubt as to the direction in which to look for an explanation of the apparent divergence between theory and observation. Everything points to a changing condition of the reflecting surface, and this suggests that a consideration of the layer of transition will show how Fresnel's laws are violated in this as in some other directions. This will be discussed further in the next chapter. Meanwhile it should be remarked that Lord Rayleigh found when dealing with reflection from liquids*, that the departure from Fresnel's formula diminished almost to zero when care was taken to remove all greasy contamination from the surface, and concluded that "there is no experimental evidence against the rigorous applicability of Fresnel's formulae to the ideal case of an abrupt transition between two uniform transparent media."

Total Reflection.

Thus far we have supposed that when a ray of light is going from one medium to another, the index of refraction of the second medium is greater than that of the first. We have now to consider the case when the media are interchanged, so that the first medium is of higher refractive index than the second. The formulae in this case can, in general, be deduced from those already obtained by replacing μ by $1/\mu$. There is one matter, however, in which the results are so completely modified as to deserve a special investigation. The relation between ϕ and ϕ', the angles of incidence and refraction, is given by Snell's law

$$\sin\phi = \frac{1}{\mu}\sin\phi'.$$

Let ϕ_1 be the angle whose sine is $1/\mu$, then since

$$\sin\phi' = \sin\phi/\sin\phi_1$$

we see that ϕ' is real or not according as ϕ is less or greater than ϕ_1. The angle ϕ_1 is called the critical angle, and if the angle of incidence be greater than this there is no refracted ray in the ordinary sense. The reflection is then said to be total.

* *Scientific Papers*, Vol. III. p. 496. *Phil. Mag.* XXXIII. pp. 1—19.

The investigation of the change of amplitude and phase in the case of total reflection follows the same lines as in the case of ordinary reflection considered above. Thus, if the displacement (ξ, η, ζ) be perpendicular to the plane of incidence, we have $\xi = 0 = \eta$, and in the first medium

$$\zeta = e^{ip\,[t-(x\cos\phi+y\sin\phi)\,\mu/c]} - re^{ip\,[t+(x\cos\phi-y\sin\phi)\,\mu/c]},$$

the first term corresponding to the incident and the second to the reflected wave. In the second medium we have

$$\zeta = se^{ip\,[t-(nx+y\sin\phi)\,\mu/c]},$$

which represents the refracted wave. In order to satisfy the fundamental dynamical equation in the second medium we must have

$$(n^2 + \sin^2\phi)\,\mu^2 = 1,$$

and so
$$n = -i\sqrt{\sin^2\phi - 1/\mu^2}.$$

The refracted wave is thus represented by

$$\zeta = se^{-\frac{p\mu x}{c}\sqrt{\sin^2\phi - 1/\mu^2}} \, . \, e^{ip(t-\mu y\sin\phi/c)}.$$

The boundary conditions require the continuity of ζ and g/μ^2, i.e. of ζ and $\dfrac{1}{\mu^2}\dfrac{\partial\zeta}{\partial x}$. From these we get

$$1 - r = s,$$

and
$$1 + r = \frac{n\mu^2 s}{\cos\phi} = -i\frac{\mu^2 s\sqrt{\sin^2\phi - 1/\mu^2}}{\cos\phi} = -is\tan\alpha,$$

where
$$\tan\alpha = \frac{\mu^2\sqrt{\sin^2\phi - 1/\mu^2}}{\cos\phi}.$$

Whence $\quad s = 2\cos\alpha \, . \, e^{i\alpha}$ and $r = -e^{2i\alpha} = e^{i(2\alpha-\pi)}$.

Thus the reflected wave has the same amplitude (and therefore the same intensity) as the incident wave, but there is a change of phase represented by $2\alpha - \pi$. The disturbance in the second medium is represented by

$$\zeta = 2q\cos\alpha \, . \, e^{i\,[p\,(t-\mu y\sin\phi/c)+\alpha]},$$

where
$$q = e^{-\frac{p\mu x}{c}\sqrt{\sin^2\phi - 1/\mu^2}} = e^{-2\pi\mu x/\lambda \, . \, \sqrt{\sin^2\phi - 1/\mu^2}},$$

λ being the wave length in the second medium. There is thus a

change of phase represented by α. The amplitude at the interface is $2\cos\alpha$, but owing to the factor q, it diminishes very rapidly as x increases, so that the disturbance practically disappears at a depth of a few wave lengths. The following table gives the values of the factor q, for different incidences, and values of x, in the case where $\mu = 1{\cdot}596$, the critical angle being $38^\circ\ 47'$.

ϕ...	39°	40°	45°	50°	60°	75°	85°
$x/\lambda = 1$	·5433	·2353	·0372	·0120	·0025	·0005	·0004
$x/\lambda = 2$	·2952	·0553	·0014	·0001	$10^{-6}\times 6$	$10^{-7}\times 3$	$10^{-7}\times 2$
$x/\lambda = 3$	·1604	·0130	$10^{-5}\times 5$	$10^{-6}\times 2$	$10^{-8}\times 1{\cdot}5$	$10^{-10}\times 1{\cdot}6$	$10^{-11}\times 7{\cdot}6$
$x/\lambda = 4$	·0873	·0031	$10^{-6}\times 2$	$10^{-8}\times 2$	$10^{-11}\times 4$	$10^{-14}\times 9$	$10^{-14}\times 3$

Similarly if the displacement (ξ, η, ζ) be parallel to the plane of incidence we have, with the notation employed in the corresponding case of ordinary reflection,

$$\zeta' = e^{ip\,[t-(x\cos\phi+y\sin\phi)\,\mu/c]} + r'e^{ip\,[t+(x\cos\phi-y\sin\phi)\,\mu/c]}$$

in the first medium, and

$$\zeta' = s'e^{ip\,[t-(nx+y\sin\phi)\,\mu/c]}$$

in the second. In order to satisfy the dynamical equations, we must give n the same value as before. The boundary conditions require ξ, η and h/μ^2 to be continuous, whence we have $1 + r' = s'$, and $\cos\phi\,(1 - r') = ns'$. These give $s' = 2\cos\alpha' e^{i\alpha'}$, and $r' = e^{2i\alpha'}$, where

$$\tan\alpha' = \frac{\sqrt{\sin^2\phi - 1/\mu^2}}{\cos\phi} = \frac{\tan\alpha}{\mu^2}.$$

The intensity of the reflected beam is equal to that of the incident; but there is a change of phase of $2\alpha'$. The disturbance in the second medium disappears, as before, at a depth of a few wave lengths.

From the formulae

$$\tan\alpha' = \frac{\tan\alpha}{\mu^2} = \frac{\sqrt{\sin^2\phi - 1/\mu^2}}{\cos\phi}$$

we see that α' is less than α, except at the critical angle where

α and α' are both zero, and at grazing incidence where they are both $\pi/2$. Since

$$r = e^{i(2\alpha - \pi)} = e^{i\rho}, \text{ and } r' = e^{i2\alpha'} = e^{i\rho'},$$

it follows that as the angle of incidence increases from the critical angle to 90°, ρ increases from $-\pi$ to 0, and ρ' from 0 to π, and their difference is always an obtuse angle. The component of the displacement perpendicular to the plane of incidence lags behind that parallel to this plane by an amount Δ' in phase, where

$$\Delta' = \rho' - \rho = \pi - 2(\alpha - \alpha').$$

The component of the curl perpendicular to the plane of incidence is in advance of that parallel to this plane by the same amount Δ'. The above formulae for α and α' give

$$\cot \Delta'/2 = \sqrt{\sin^2\phi - 1/\mu^2}/\sin\phi \tan\phi.$$

The following table gives the values of α, α' and Δ' for various incidences beyond the critical angle, for the case mentioned above where $\mu = 1{\cdot}596$ and the critical angle is 38° 47′.

ϕ...	39°	40°	45°	50°	60°	75°	85°
α	11° 17′	25° 38′	49° 46′	60° 14′	71° 50′	82° 17′	87° 28′
α'	4° 29′	10° 40′	24° 53′	34° 27′	50° 6′	70° 57′	83° 35′
Δ'	166° 24′	150° 4′	130° 14′	128° 26′	136° 32′	157° 20′	172° 14′

These results are exhibited in Fig. 17 below, which represents the march of Δ' as the angle of incidence increases from normal to grazing incidence. *A* corresponds to the polarising angle and *C* to the critical angle.

As Δ' is equal to π at the critical angle and at grazing incidence, it will have a minimum value somewhere within the range of total reflection, as appears from the figure below. The position of this minimum is found by making Δ' stationary, so that

$$\frac{d\alpha}{d\phi} = \frac{d\alpha'}{d\phi'}.$$

This gives

$$\tan^2\alpha = \mu^2 = \mu^4 \cdot \frac{\sin^2\phi - 1/\mu^2}{1 - \sin^2\phi},$$

whence

$$\sin^2\phi = 2/(\mu^2 + 1).$$

At this angle we have

$$\tan\alpha = \mu, \quad \text{and} \quad \tan\alpha' = 1/\mu,$$

so that

$$\alpha + \alpha' = \pi/2, \quad \text{and} \quad \Delta' = 4\alpha'.$$

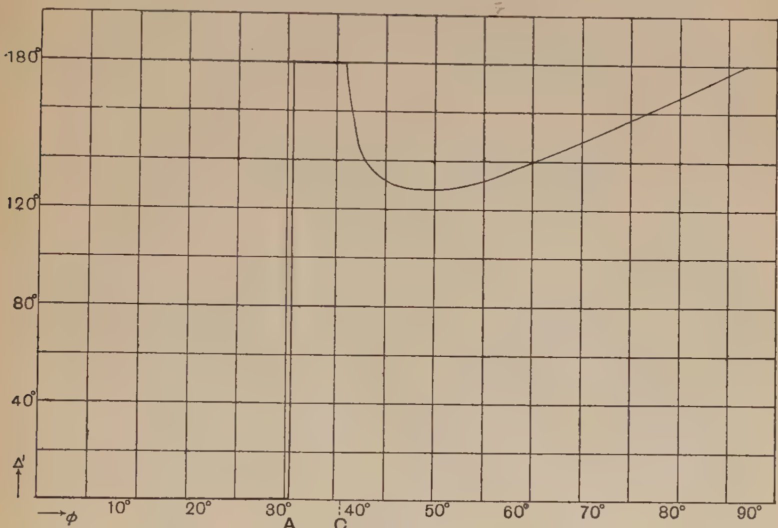

Fig. 17.

Thus at the angle for which Δ' is a minimum, α' is Brewster's angle $\tan^{-1} 1/\mu$. The following table gives the values of these quantities for various refractive indices.

μ	α' (Brewster's angle)	ϕ_1 (critical angle)	ϕ (angle for which Δ' is least)	Least value of $\Delta' = 4\alpha'$
1·46	34° 24′	43° 14′	53° 3′	137° 38′
1·50	33° 41′	41° 49′	51° 40′	134° 45′
1·51	33° 31′	41° 28′	51° 20′	134° 3′
1·596	32° 4′	38° 49′	48° 40′	128° 16′
1·60	32° 0′	38° 41′	48° 33′	128° 1′
2·371	22° 52′	24° 57′	33° 20′	91° 28′
2·414	22° 30′	24° 28′	32° 46′	90°
2·434	22° 20′	24° 15′	32° 31′	89° 20′
2·454	22° 10′	24° 3′	32° 15′	88° 40′

The minimum value of Δ' being $4\alpha'$, we see that if this is $\pi/2$ we must have

$$\alpha' = \pi/8, \text{ and } \mu = \cot \alpha' = 2{\cdot}414.$$

The angle of incidence for which the difference of phase is $\pi/2$ is called the Principal Incidence. Below the critical angle this coincides with the polarising angle. Within the limits of total reflection there will be no Principal Incidence if μ be less than 2·414 and two principal incidences for substances such as diamond and realgar for which μ is greater than 2·414. Thus we cannot obtain a phase difference of $\pi/2$ by means of total reflection at a single surface for substances whose refractive indices are less than 2·414. This, however, can be effected by arranging, as is done in Fresnel's rhomb, for two reflections with a phase difference of $3\pi/4$ at each reflection, since a retardation of $3\pi/2$ is virtually the same as an acceleration of $\pi/2$. Fig. 17 above shows that, as a rule, there will be two angles of incidence for which the difference of phase is $3\pi/4$, and it is easy to obtain a formula for their magnitudes. Since

$$\Delta' = \pi - 2(\alpha - \alpha'),$$

we have

$$\tan \frac{\Delta'}{2} = \cot(\alpha - \alpha') = \frac{\sin \phi \tan \phi}{\sqrt{\sin^2 \phi - 1/\mu^2}}.$$

Putting $\Delta' = 3\pi/4$, this gives

$$\mu^2 \sin^4 \phi (4 - 2\sqrt{2}) - (\mu^2 + 1) \sin^2 \phi + 1 = 0,$$

a quadratic equation in $\sin^2 \phi$, which has real roots provided μ be greater than 1·496. The following table gives the two values of ϕ for different kinds of glass, the refractive index being μ.

μ	1·5	1·51	1·55	1·596	1·6
Smaller angle of incidence	50° 13′ 46″	48° 37′ 27″	45° 14′ 29″	42° 41′ 15″	42° 29′ 45″
Larger angle	53° 15′ 31″	54° 37′ 23″	57° 4′ 42″	58° 37′ 35″	58° 44′ 2″

Fresnel's rhomb is a rhomb of glass, a principal section of which is shown in Fig. 18. The angles of the rhomb are chosen so that a ray incident at right angles to one face, suffers total reflection at

two parallel faces, and emerges normally to the face opposite that at which it enters. The difference of phase produced by passing

Fig. 18.

through the rhomb being $\pi/2$, it is evident that the acute angle of the rhomb must be one or other of the two values of ϕ determined above. Of these two possible values the larger is to be preferred, because with it the errors due to slight changes in μ for different colours are less than with the smaller value of ϕ, and also because a small deviation from ϕ will introduce less error when ϕ is large than an equal deviation when ϕ is small. For this latter reason also, the refractive index should exceed 1·496 by as small a quantity as possible. If the light incident at A were plane polarised at an angle of 45° with the plane of incidence, so that the amplitudes of the displacements parallel and perpendicular to this plane were equal (and we have seen that natural light could be so represented), then the light on emerging from the rhomb would be circularly polarised, for the amplitudes of the displacements parallel and perpendicular to the plane of incidence would be equal, and their difference of phase would be $\pi/2$.

The following table gives the values of Δ', expressed as a fraction of the half wave length, for total reflection from a substance whose refractive index is 1·619 and critical angle 38° 9′,

ϕ...	38° 13′	39° 58′	41° 59′	44° 2′	46° 4′	47° 54′	49° 58′	51° 57′	53° 58′	55° 57′
′(Theory)	·960	·799	·744	·718	·707	·705	·707	·712	·721	·731
′ (Exp.)	·978	·813	·754	·728	·718	·712	·715	·719	·726	·730
ifference	·018	·014	·010	·010	·011	·007	·008	·007	·005	·001

and compares them with the results of experiment. Fig. 19 represents the same results graphically.

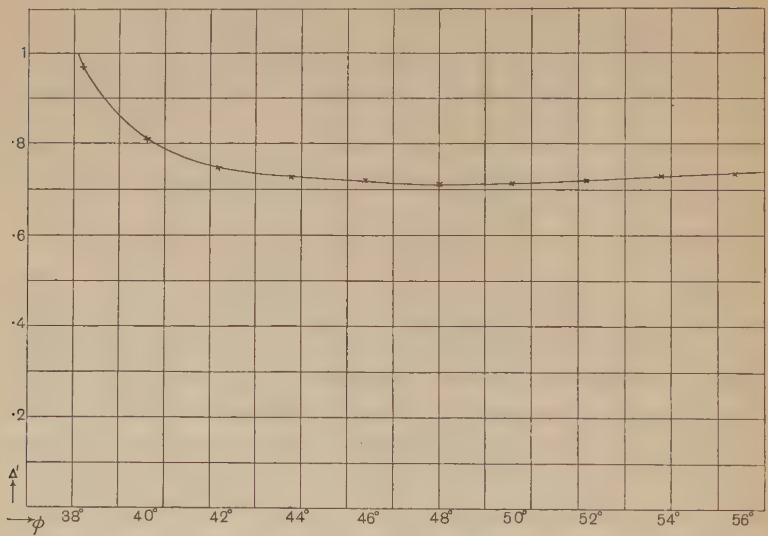

Fig. 19.

CHAPTER IV.

INFLUENCE OF A LAYER OF TRANSITION ON REFLECTION AND REFRACTION IN TRANSPARENT ISOTROPIC MEDIA.

IT appears from the last chapter that the theory of an abrupt transition from one medium to another leads to expressions for the amplitude and phase of the displacement in the reflected and refracted waves that cannot be regarded as better than a close approximation to the facts as derived from careful experiments. As a large number of such experiments indicate that the optical effects of a reflecting or refracting body depend on the condition of its surface, such as its freedom from tarnish or the state of its polish, it is clear that we must, in our analysis, take some account of these surface conditions. We shall, therefore, in the present chapter, investigate to what extent the previous formulae must be modified when the refractive index, instead of changing abruptly, changes continuously throughout a very thin layer of transition between the two media. As the law under which the refractive index changes within the layer will vary with the condition of the surface, we shall not at the outset make any hypothesis as to the nature of this law.

In dealing with the problem of reflection and refraction we shall use the same notation as in the last chapter and shall suppose that the layer of transition extends from $x=0$ to $x=d$, and is continuous, as regards the refractive index μ, with the media bounding it. It will be convenient to put

$$\sin\phi = \mu \sin\phi' = \nu\,; \quad \cos\phi = \kappa\,; \quad \mu\cos\phi' = \kappa'.$$

The dynamical equations and boundary conditions for the

layer may be obtained as before from the Principle of Action, by taking

$$T = \tfrac{1}{2}\int (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)\, d\tau, \qquad W = \tfrac{1}{2}\int \frac{c^2}{\mu^2}(f^2 + g^2 + h^2)\, d\tau;$$

but it must be remembered that μ is no longer a constant. The term containing $\delta\xi$ in the variation of W is

$$\int \frac{c^2}{\mu^2}\left(g\frac{\partial\delta\xi}{\partial z} - h\frac{\partial\delta\xi}{\partial y}\right) d\tau$$

$$= \int \frac{c^2}{\mu^2}(ng - mh)\,\delta\xi\, d\zeta - c^2\int \left[\frac{\partial}{\partial z}\left(\frac{g}{\mu^2}\right) - \frac{\partial}{\partial y}\left(\frac{h}{\mu^2}\right)\right]\delta\xi\, d\tau.$$

Hence the dynamical equations consist of

$$\ddot{\xi} = c^2\left[\frac{\partial}{\partial z}\left(\frac{g}{\mu^2}\right) - \frac{\partial}{\partial y}\left(\frac{h}{\mu^2}\right)\right]$$

and two similar equations. Also, since μ is continuous at the boundaries of the layer, the boundary conditions are satisfied by the continuity of $ng - mh$, $lh - nf$, and $mf - lg$, i.e. of g and h for surfaces parallel to the plane $x = 0$, coupled, of course, with the continuity of the displacement (ξ, η, ζ).

If the incident light be polarised at right angles to the plane of incidence we have $\xi = 0 = \eta$, while ζ is given in the first and second media by the formulae on p. 39. Within the layer itself we may take $\zeta = ue^{ip(t - \nu y/c)}$, where u is a function of x only. This gives

$$f = \frac{\partial\zeta}{\partial y} = -\frac{ip\nu}{c}\, ue^{ip(t-\nu y/c)}, \text{ and } g = -\frac{\partial\zeta}{\partial x} = -\frac{du}{dx}e^{ip(t-\nu y/c)}.$$

Substituting in the dynamical equation

$$\ddot{\zeta} = c^2\left[\frac{\partial}{\partial y}\left(\frac{f}{\mu^2}\right) - \frac{\partial}{\partial x}\left(\frac{g}{\mu^2}\right)\right],$$

we get

$$\frac{d}{dx}\left(\frac{1}{\mu^2}\frac{du}{dx}\right) + \frac{p^2}{c^2}\left(1 - \frac{\nu^2}{\mu^2}\right) u = 0.$$

Putting $x/d = x_1$ and $d_1 = pd/c = 2\pi d/\lambda$, where λ is the wave length in the first medium, we get

$$\frac{d}{dx_1}\left(\frac{1}{\mu^2}\frac{du}{dx_1}\right) + d_1^2\left(1 - \frac{\nu^2}{\mu^2}\right) u = 0.$$

This equation, of course, cannot be solved completely until we know μ^2 as a function of x_1, i.e. until we know the law of variation of μ in the layer. However, in all cases to which we shall apply the solution, d_1 is a small quantity (as will appear later), and we can thus solve the above equation by successive approximations.

A first approximation is obtained by neglecting d_1^2, when the equation becomes

$$\frac{d}{dx_1}\left(\frac{1}{\mu^2}\frac{du}{dx_1}\right)=0, \quad \text{whence} \quad \frac{1}{\mu^2}\frac{du}{dx_1}=\beta,$$

and
$$u=A+B\int_0^{x_1}\mu^2 dx_1=A+B\mu^2 M,$$

where M is a function of x_1. As the next approximation we put

$$u=A+B\mu^2 M+d_1^2 v,$$

and the equation for v is

$$\frac{d}{dx_1}\left(\frac{1}{\mu^2}\frac{dv}{dx_1}\right)+\left(1-\frac{\nu^2}{\mu^2}\right)(A+B\mu^2 M)=0.$$

Integrating this, we get

$$\frac{dv}{dx_1}+A\mu^2 x_1+B\mu^4 M_1-A\nu^2 M_3-B\nu^2\mu^2 M_4=0,$$

where

$$\mu^2 M_1=\int_0^{x_1}\mu^2 M dx_1, \quad \frac{M_3}{\mu^2}=\int_0^{x_1}\frac{1}{\mu^2}dx_1, \quad M_4=\int_0^{x_1}M dx_1.$$

A second integration then gives

$$v+A\mu^2(x_1 M-M_1)+B\mu^4 M_5-A\nu^2 M_6-B\nu^2\mu^2 M_7=0,$$

where

$$\mu^4 M_5=\int_0^{x_1}\mu^4 M_1 dx_1, \quad M_6=\int_0^{x_1}M_3 dx_1, \quad \mu^2 M_7=\int_0^{x_1}\mu^2 M_4 dx_1.$$

When $x_1=0$ all the M's are zero, and when $x_1=1$ we have

$$M=E,\ M_1=H,\ M_3=F,\ M_4=J,\ M_5=K,\ M_6=L,\ M_7=N$$

where $E, \ldots, N$ are constants depending on the law of variation of μ within the layer.

Thus when $x_1=0$ we have $u=A$, $du/dx_1=B$, and when $x_1=1$ we have similarly

$$u=A+B\mu^2 E+d_1^2[A\{\mu^2(H-E)+\nu^2 L\}+B\mu^2(\nu^2 N-\mu^2 K)],$$

and
$$du/dx_1=B\mu^2+d_1^2[A(\nu^2 F-\mu^2)+B\mu^2(\nu^2 J-\mu^2 H)].$$

The boundary conditions give

$$A = 1 - r,\quad B = -id_1\kappa(1+r),$$

$$s = A + B\mu^2 E + d_1^2[A\{\mu^2(H-E) + \nu^2 L\} + B\mu^2(\nu^2 N - \mu^2 K)],$$

$$-i\kappa' s = B\mu^2/d_1 + d_1[A(\nu^2 F - \mu^2) + B\mu^2(\nu^2 J - \mu^2 H)].$$

On eliminating A and B from these equations, we get two equations to determine r and s, which give the amplitude and phase of the reflected and refracted beams. If $r = Re^{i\rho}$ and $s = Se^{i\sigma}$, then R and S are the amplitudes, and ρ and σ the phases, of the displacements in the reflected and refracted waves.

In dealing with the above equations we shall, as the first approximation, neglect squares and higher powers of d_1. The equations then yield

$$s + r[1 + id_1\kappa E\mu^2] = 1 - id_1\kappa E\mu^2$$

and $$\kappa' s - r[\kappa\mu^2 - id_1(\nu^2 F - \mu^2)] = \kappa\mu^2 + id_1(\nu^2 F - \mu^2).$$

From these we obtain

$$s = \frac{[1 - id_1\kappa E\mu^2][\kappa\mu^2 - id_1(\nu^2 F - \mu^2)] + [1 + id_1\kappa E\mu^2][\kappa\mu^2 + id_1(\nu^2 F - \mu^2)]}{\kappa'[1 + id_1\kappa E\mu^2] + \kappa\mu^2 - id_1(\nu^2 F - \mu^2)}$$

$$= 2\kappa\mu^2[(\kappa' + \kappa\mu^2) + id_1(\kappa\kappa' E\mu^2 + \mu^2 - \nu^2 F)]^{-1}, \text{ to our order,}$$

$$= Se^{i\sigma},$$

where

$$S = \frac{2\kappa\mu^2}{\kappa' + \kappa\mu^2} = \frac{2\mu^2\cos\phi}{\mu\cos\phi' + \mu^2\cos\phi} = \frac{\sin 2\phi}{\sin(\phi+\phi')\cos(\phi-\phi')},$$

and $$\tan\sigma = -\frac{\kappa\kappa' E\mu^2 + \mu^2 - \nu^2 F}{\kappa' + \kappa\mu^2}\,.\,d_1.$$

Hence, to this order, the amplitude of the refracted wave is the same as in the ideal case of an abrupt transition; but there is a *small* change of phase.

Similarly for the reflected wave we get

$$r = -\frac{\kappa\mu^2 - \kappa' + id_1(\kappa\kappa' E\mu^2 - \mu^2 + \nu^2 F)}{\kappa\mu^2 + \kappa' + id_1(\kappa\kappa' E\mu^2 + \mu^2 - \nu^2 F)} = Re^{i\rho},$$

where $$R = \pm\frac{\kappa\mu^2 - \kappa'}{\kappa\mu^2 + \kappa'} = \pm\frac{\tan(\phi - \phi')}{\tan(\phi + \phi')},$$

the sign being chosen to make R positive, and

$$\tan\rho = -\frac{2\kappa\mu^2\,[E\kappa'^2 + F\nu^2 - \mu^2]}{\kappa'^2 - (\kappa\mu^2)^2} \, . \, d_1$$

$$= -\frac{2\mu^2\cos\phi\,[(F-E)\sin^2\phi - (1-E)\,\mu^2]}{(\mu^2-1)\,[\sin^2\phi - \mu^2\cos^2\phi]} \, . \, d_1,$$

except in the neighbourhood of Brewster's angle, where

$$\kappa\mu^2 - \kappa' = 0,$$

and we have

$$R = \pm\frac{\kappa\kappa' E\mu^2 - \mu^2 - \nu^2 F}{\kappa\mu^2 + \kappa'} \, . \, d_1 = \pm\frac{d_1}{2\sqrt{1+\mu^2}}\,[E\mu^2 + F - (1+\mu^2)]$$

$$= \pm\frac{d_1}{2\sqrt{1+\mu^2}}\, I,$$

where I denotes the integral

$$\int_0^1 \frac{(\mu'^2 - 1)\,(\mu'^2 - \mu^2)}{\mu'^2}\, dx_1,$$

μ' being the refractive index in the layer.

In this neighbourhood also $\tan\rho$ is infinite, so that

$$\rho = \pm\frac{\pi}{2}.$$

From these formulae we see that with reflection as well as with refraction the amplitude is, in general, given correctly, to our order of approximation, by Fresnel's formulae, but that there is a small change of phase depending on d_1. The only region in which there is a marked departure from Fresnel's formulae is in the neighbourhood of Brewster's angle. Here R does not vanish, although it is very small, and there is a change of phase of a quarter wave length.

These results have been obtained by neglecting squares of d_1. For some purposes it is necessary to proceed to a higher order of approximation and retain terms in d_1^2. The equations for r and s then become

$$s + r\,[1 + id_1\kappa E\mu^2 + d_1^2\,\{\mu^2(H-E) + \nu^2 L\}]$$
$$= 1 - id_1\kappa E\mu^2 + d_1^2\,\{\mu^2(H-E) + \nu^2 L\},$$

$$\kappa' s - r\,[\kappa\mu^2 - id_1(\nu^2 F - \mu^2) + d_1^2\kappa\mu^2(\nu^2 J - \mu^2 H)]$$
$$= \kappa\mu^2 + id_1(\nu^2 F - \mu^2) + d_1^2\kappa\mu^2(\nu^2 J - \mu^2 H).$$

Whence we get

$$s=\frac{2\kappa\mu^2\left[1+d_1^2\left\{\nu^2(L+J+EH)-2E\mu^2\right\}\right]}{(\kappa'+\kappa\mu^2)+id_1(\kappa\kappa' E\mu^2+\mu^2-\nu^2F)+d_1^2\left[\kappa\mu^2(\nu^2J-\mu^2H)+\kappa'\left\{\mu^2(H-E)+\nu^2L\right\}\right]}$$

$$=Se^{i\sigma},$$

where $$S=\frac{2\kappa\mu^2}{\kappa'+\kappa\mu^2}\left[1+d_1^2\left\{\nu^2(L+J+EH)\right.\right.$$

$$\left.\left.-\frac{(\kappa\kappa' E\mu^2+\mu^2-\nu^2F)^2+2(\kappa'+\kappa\mu^2)\left\{\kappa'(\mu^2\overline{H-E}+\nu^2L)+\kappa\mu^2(\nu^2J-\mu^2H)\right\}}{2(\kappa'+\kappa\mu^2)^2}\right\}\right],$$

and $$\tan\sigma=-\frac{\kappa\kappa' E\mu^2+\mu^2-\nu^2F}{\kappa'+\kappa\mu^2}\,.\,d_1.$$

Similarly

$$r=-\frac{\kappa\mu^2-\kappa'+id_1(\kappa\kappa' E\mu^2-\mu^2+\nu^2F)+d_1^2\left[\kappa\mu^2(\nu^2J-\mu^2H)-\kappa'\left\{\mu^2(H-E)+\nu^2L\right\}\right]}{\kappa\mu^2+\kappa'+id_1(\kappa\kappa' E\mu^2+\mu^2-\nu^2F)+d_1^2\left[\kappa\mu^2(\nu^2J-\mu^2H)+\kappa'\left\{\mu^2(H-E)+\nu^2L\right\}\right]}$$

$$=Re^{i\rho},$$

where

$$R=\pm\frac{\kappa\mu^2-\kappa'}{\kappa\mu^2+\kappa'}\left[1+\frac{2\kappa\kappa'\mu^2d_1^2}{\left\{(\kappa\mu^2)^2-\kappa'^2\right\}^2}\left\{(\kappa\kappa' E\mu^2)^2+(\mu^2-\nu^2F)^2\right.\right.$$

$$\left.\left.-E(\mu^2-\nu^2F)(\kappa^2\mu^4+\kappa'^2)+(\kappa^2\mu^4-\kappa'^2)(\nu^2J-H+\mu^2\overline{E-2H})\right\}\right],$$

and $$\tan\rho=-\frac{2\kappa\mu^2\left[E\kappa'^2+F\nu^2-\mu^2\right]}{\kappa'^2-\kappa^2\mu^4}\,.\,d_1,$$

except at Brewster's angle, where

$$R=\pm\frac{\kappa\kappa' E\mu^2-\mu^2+\nu^2F}{\kappa\mu^2+\kappa'}\,.\,d_1,\ \text{and}\ \ \rho=\pm\left(\frac{\pi}{2}-\theta\right),$$

where $\tan\theta$

$$=\frac{\left[\kappa\mu^2(\nu^2J-\mu^2H)-\kappa'\left\{\mu^2(H-E)+\nu^2L\right\}\right](\kappa'+\kappa\mu^2)+\left[(\kappa\kappa' E\mu^2)^2-(\mu^2-\nu^2F)^2\right]}{(\kappa'+\kappa\mu^2)(\kappa\kappa' E\mu^2-\mu^2+\nu^2F)}\,.\,d_1$$

$$=\frac{d_1}{\sqrt{1+\mu^2}}\cdot\frac{2\mu^2\left[(J-2H-L)+\mu^2(E-2H)\right]+E^2\mu^4-(\mu^2+1-F)^2}{E\mu^2-(\mu^2+1-F)}.$$

We must next consider the case in which the incident light is polarised parallel to the plane of incidence. We then have $\xi'=0=\eta'$, while ζ' in the first and second media is given by the

formulae on p. 40. Within the layer $\zeta' = ue^{ip(t-\nu y/c)}$, where u is a function of x only. This gives

$$\xi = \frac{\partial \zeta'}{\partial y} = -\frac{ip\nu}{c} . ue^{ip(t-\nu y/c)}, \quad \eta = -\frac{\partial \zeta'}{\partial x} = -\frac{du}{dx} e^{ip(t-\nu y/c)};$$

$$\zeta = 0, \quad f = 0 = g,$$

$$h = \frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y} = -\nabla^2 \zeta' = \left[\frac{p^2\nu^2}{c^2} u - \frac{d^2u}{dx^2}\right] e^{ip(t-\nu y/c)}.$$

We have seen above that the dynamical equations are

$$\ddot{\xi} = c^2 \left[\frac{\partial}{\partial z}\left(\frac{g}{\mu^2}\right) - \frac{\partial}{\partial y}\left(\frac{h}{\mu^2}\right)\right],$$

and two similar equations. These are all satisfied if

$$\ddot{\zeta}' = -\frac{c^2}{\mu^2} h = \frac{c^2}{\mu^2} \nabla^2 \zeta'.$$

Hence we require

$$\frac{d^2u}{dx^2} + \frac{p^2}{c^2}(\mu^2 - \nu^2) = 0,$$

or
$$\frac{d^2u}{dx_1^2} + d_1^2 (\mu^2 - \nu^2) u = 0.$$

Solving this by successive approximations, we get in the first place $u = A + Bx_1$, and as the second approximation

$$u = A + Bx_1 + d_1^2 v,$$

where
$$d^2v/dx_1^2 + (\mu^2 - \nu^2)(A + Bx_1) = 0.$$

Integrating this equation, and making use of the relation

$$\int x_1 \mu^2 dx_1 = x_1 \mu^2 M - \int \mu^2 M dx_1 = \mu^2 (x_1 M - M_1),$$

we get

$$\frac{dv}{dx_1} = \nu^2 (Ax_1 + \tfrac{1}{2}Bx_1^2) - \mu^2 (AM + BMx_1 - BM_1),$$

and
$$v = \nu^2 (\tfrac{1}{2}Ax_1^2 + \tfrac{1}{6}Bx_1^3) - \mu^2 (AM_1 + BM_1x_1 - 2BM_2),$$

where
$$\mu^2 M_2 = \int_0^{x_1} \mu^2 M_1 dx_1.$$

When $x_1 = 0$, $M_2 = 0$, and when $x_1 = 1$, $M_2 = G$. Thus we have $u = A$ and $du/dx_1 = B$ when $x_1 = 0$, and when $x_1 = 1$,

$$u = A + B + d_1^2 [\nu^2 (A/2 + B/6) - \mu_2^2 \{H(A + B) - 2BG\}]$$

and
$$du/dx_1 = B + d_1^2 [\nu^2 (A + B/2) - \mu^2 \{E(A + B) - BH\}].$$

The boundary conditions give

$$A = 1 + r', \quad B = -id_1\kappa(1 - r'),$$

$$s'/\mu = A + B + d_1^2[\nu^2(A/2 + B/6) - \mu^2\{H(A + B) - 2BG\}],$$

$$-id_1\kappa' s'/\mu = B + d_1^2[\nu^2(A + B/2) - \mu^2\{E(A + B) - BH\}],$$

from which r' and s' can be determined, by eliminating A and B.

As the first approximation we shall retain only the first power of d_1, and we then get $s'/\mu - r'(1 + id_1\kappa) = 1 - id_1\kappa$,

and $\qquad \kappa' s'/\mu + r'\{\kappa + id_1(E\mu^2 - \nu^2)\} = \kappa - id_1(E\mu^2 - \nu^2).$

Thus

$$s' = \mu \cdot \frac{\{1 - id_1\kappa\}\{\kappa + id_1(E\mu^2 - \nu^2)\} + \{1 + id_1\kappa\}\{\kappa - id_1(E\mu^2 - \nu^2)\}}{\kappa + \kappa' + id_1(\kappa\kappa' + E\mu^2 - \nu^2)}$$

$$= \frac{2\mu\kappa}{\kappa + \kappa' + id_1(\kappa\kappa' + E\mu^2 - \nu^2)} = S' e^{i\sigma'},$$

where

$$S' = \frac{2\mu\kappa}{\kappa + \kappa'} = \frac{\sin 2\phi}{\sin(\phi + \phi')}, \text{ and } \tan\sigma' = -\frac{\kappa\kappa' + E\mu^2 - \nu^2}{\kappa + \kappa'} \cdot d_1.$$

Similarly $\qquad r' = \dfrac{\kappa - \kappa' + id_1(\kappa\kappa' - E\mu^2 + \nu^2)}{\kappa + \kappa' + id_1(\kappa\kappa' + E\mu^2 - \nu^2)} = R' e^{i\rho'},$

where

$$R' = \frac{\kappa' - \kappa}{\kappa' + \kappa} = \frac{\sin(\phi - \phi')}{\sin(\phi + \phi')}, \text{ and } \tan\rho' = -\frac{2\kappa\mu^2(1 - E)}{\mu^2 - 1} \cdot d_1.$$

Hence, both in the reflected and in the refracted wave, the amplitude is the same as that given by Fresnel's formula, but there is a small change of phase.

If we proceed to a higher order of approximation and retain squares of d_1 we get similarly

$$s' = \frac{2\mu\kappa}{\kappa + \kappa' + id_1(\kappa\kappa' + E\mu^2 - \nu^2) + d_1^2[(\kappa + \kappa')\nu^2/2 - \mu^2\{E\kappa + H(\kappa' - \kappa)\}]} = S' e^{i\sigma'},$$

where

$$S' = \frac{2\mu\kappa}{\kappa + \kappa'}\left[1 - \frac{d_1^2}{2(\kappa + \kappa')^2}\{(\kappa\kappa' + E\mu^2 - \nu^2)^2 + (\kappa + \kappa')^2\nu^2 - 2\mu^2(\kappa + \kappa')(E\kappa + H\kappa' - H\kappa)\}\right],$$

and $$\tan\sigma' = -\frac{\kappa\kappa' + E\mu^2 - \nu^2}{\kappa + \kappa'} \cdot d_1;$$

$$r' = \frac{\kappa - \kappa' + id_1(\kappa\kappa' - E\mu^2 + \nu^2) + d_1^2[(\kappa - \kappa')\nu^2/2 + \mu^2(E\kappa - H\kappa' + H\kappa)]}{\kappa + \kappa' + id_1(\kappa\kappa' + E\mu^2 - \nu^2) + d_1^2[(\kappa + \kappa')\nu^2/2 - \mu^2(E\kappa + H\kappa' - H\kappa)]} = R' e^{i\rho'},$$

where

$$R' = \frac{\kappa' - \kappa}{\kappa' + \kappa}\left[1 + \frac{2\kappa\kappa' d_1^2}{(\kappa'^2 - \kappa^2)^2}\{\kappa^2\kappa'^2 + (E\mu^2 - \nu^2)^2 + \mu^2(E - 2H)(\kappa'^2 - \kappa^2) - (E\mu^2 - \nu^2)(\kappa'^2 + \kappa^2)\}\right],$$

and

$$\tan\rho' = -\frac{2\kappa\mu^2(1-E)}{\mu^2 - 1}\,.\,d_1.$$

It will be observed that the change of phase (ρ' and σ') is the same as in the first approximation.

The quantities that can be most accurately measured experimentally are ϵ and Δ, the ratio of the amplitudes and the difference of phase for the displacements perpendicular and parallel respectively to the plane of incidence. These are given in terms of the quantities dealt with above, by means of the relation

$$\epsilon e^{i\Delta} = r/r' = R/R'\,.\,e^{i(\rho - \rho')},$$

and from what we have proved, it appears that it is only in the neighbourhood of Brewster's angle that there is any appreciable ± departure from the values given for ϵ and Δ by Fresnel's formulae. The value of ϵ at Brewster's angle is called the coefficient of ellipticity and will be denoted by ϵ_1. Neglecting d_1 we have, at Brewster's angle,

$$R = \pm\frac{d_1 I}{2\sqrt{1+\mu^2}}, \text{ and } R' = \frac{\sin(\phi - \phi')}{\sin(\phi + \phi')} = \frac{\mu^2 - 1}{\mu^2 + 1}.$$

Hence

$$\epsilon_1 = \pm\frac{I d_1\sqrt{1+\mu^2}}{2(\mu^2 - 1)}.$$

Since

$$I = \int_0^1 \frac{(\mu'^2 - 1)(\mu'^2 - \mu^2)\,dx}{\mu'^2},$$

we see that if μ' lies between 1 and μ, I is negative, so that we must take the lower sign with ϵ_1, which is essentially positive. If we write ϵ_1 in the form

$$\epsilon_1 = \pm\frac{d_1\sqrt{1+\mu^2}}{2(\mu^2 - 1)}\int_0^1\left[\mu'^2 + \frac{\mu^2}{\mu'^2} - (1 + \mu^2)\right]dx_1,$$

we see that ϵ_1 is a maximum along with $\mu'^2 + \mu^2/\mu'^2$, i.e. when $\mu'^2 = \mu$. If μ'^2 had this value throughout the layer we should have

$$\epsilon_1 = \frac{d_1}{2}\,\frac{\mu - 1}{\mu + 1}\sqrt{1+\mu^2} = \frac{\pi d}{\lambda}\,\frac{\mu - 1}{\mu + 1}\sqrt{1+\mu^2}.$$

This enables us to obtain an upper limit for the thickness of the layer necessary to produce the ellipticity observed at Brewster's angle in any given case. Thus with Kurz's experiments on reflection from glass for which $\mu = 1{\cdot}5963$ and $\epsilon_1 = 0{\cdot}0344$, we have $d/\lambda = 0{\cdot}0025$ and $d_1 = 0{\cdot}0159$; while with Jaurin's experiments on reflection from diamond for which $\mu = 2{\cdot}434$ and $\epsilon_1 = 0{\cdot}0241$, we have $d/\lambda = 0{\cdot}0074$ and $d_1 = 0{\cdot}0464$.

Since ϵ_1 can be readily determined from experiment, it is convenient to express ϵ and Δ in terms of this quantity. Thus we have

$$\epsilon = \frac{R}{R'} = \frac{\sin(\phi + \phi')}{\sin(\phi - \phi')} \frac{\sqrt{(\cos\phi - \cos\phi'/\mu)^2 + 4\epsilon_1^2 (\mu^2 - 1)^2/(\mu^2 + 1)^3}}{\cos\phi + \cos\phi'/\mu},$$

which is practically equivalent to Fresnel's formula

$$\epsilon = \pm \cos(\phi + \phi')/\cos(\phi - \phi')$$

outside the neighbourhood of Brewster's angle. The following table gives the values of ϵ^2 for different angles of incidence near Brewster's angle and compares them with the results of experiment. The reflection is from diamond for which $\mu = 2{\cdot}434$,

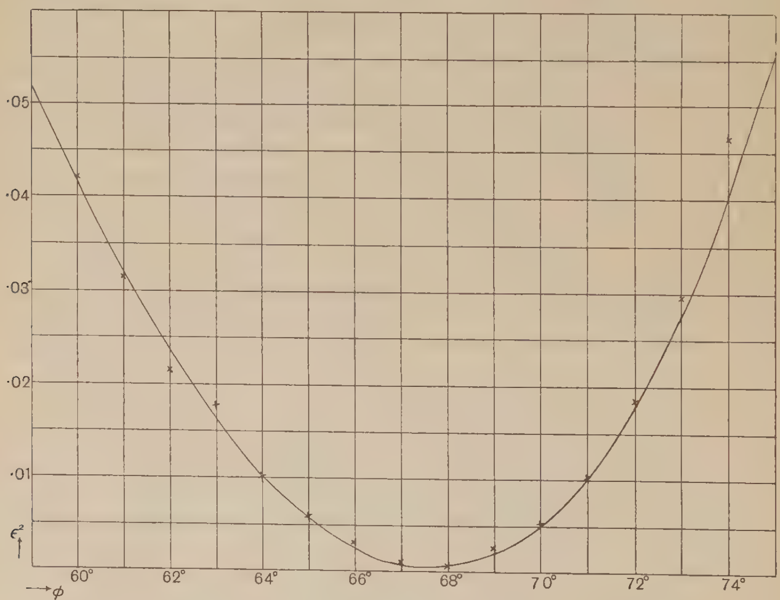

Fig. 20.

Brewster's angle is 67° 40′, and the coefficient of ellipticity observed was 0·0241.

ϕ......	66°	66° 30′	67°	67° 30′	68°	68° 30′	69°
ϵ^2 (Theory)	·00282	·00168	·00092	·00061	·00068	·00126	·00200
ϵ^2 (Experiment)	·00303	·00173	·00092	·00057	·00068	·00128	·00265
Difference	– ·00021	– ·00005	0	+ ·00004	0	– ·00002	– ·00065

We have already seen (p. 42) that outside of this region Fresnel's formula gives an admirable representation of the facts. These results are exhibited in Fig. 20.

For Δ we have approximately,

$$\tan \Delta = \tan \rho - \tan \rho' = -2\kappa\mu^2 d_1 \left[\frac{E\kappa'^2 + F\nu^2 - \mu^2}{\kappa'^2 - \kappa^2\mu^4} - \frac{1-E}{\mu^2 - 1}\right]$$

$$= -\frac{2\mu^2 d_1}{\mu^2 - 1}[(F-1) - \mu^2(1-E)]\frac{\sin\phi\tan\phi}{\tan^2\phi - \mu^2}$$

$$= \frac{4\epsilon_1\mu^2}{\sqrt{1+\mu^2}} \cdot \frac{\sin\phi\tan\phi}{\tan^2\phi - \mu^2}.$$

The values of Δ deduced from this formula for the case of reflection from diamond mentioned above are given in the next table, and the results compared with those observed by Jaurin.

ϕ.....,	60°	61°	62°	63°	64°	65°	66° 30′	67°	67° 30′
– Δ (Theory)	·033	·038	·045	·055	·070	·096	·197	·282	·430
– Δ (Exp.)	·033	·042	·047	·063	·073	·105	·202	·288	·437
Difference	0	– ·004	– ·002	– ·008	– ·003	– ·009	– ·005	– ·006	– ·007
ϕ......	68°	68° 30′	69°	70°	71°	72°	73°	74°	75°
– Δ (Theory)	·618	·749	·822	·891	·923	·941	·952	·961	·967
– Δ (Exp.)	·640	·769	·826	·897	·928	·940	·948	·955	·962
Difference	– ·022	– ·020	– ·004	– ·006	– ·005	+ ·001	+ ·004	+ ·006	+ ·005

Fig. 21 shows the march of Δ graphically, Δ being expressed throughout as a fraction of the half wave length.

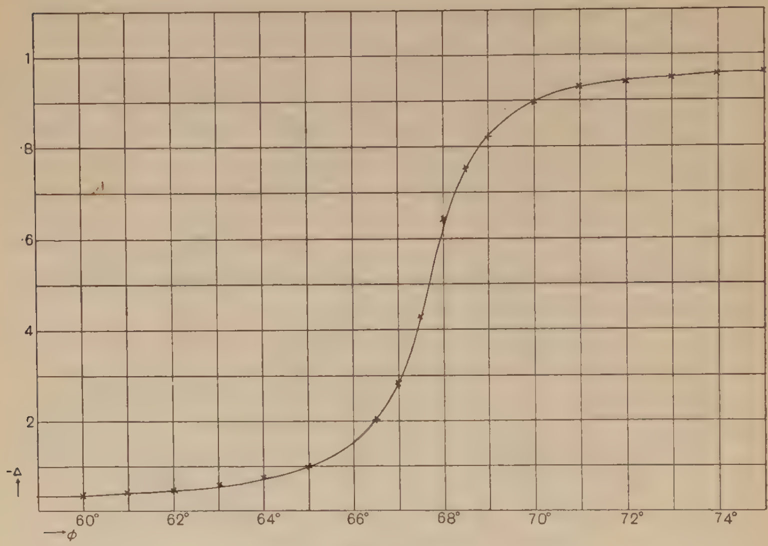

Fig. 21.

If we are dealing with light polarised at right angles to the plane of incidence we have $r = Re^{i\rho} = x + iy$, say, and in expressing the various terms of these formulae it will be convenient to introduce the coefficient of ellipticity ϵ_1. Thus we have

$$\tan\rho = -\frac{2\kappa\mu^2 d_1(E\kappa'^2 + F\nu^2 - \mu^2)}{\kappa'^2 - \kappa^2\mu^4} = \frac{4\epsilon_1\mu^4\sec\phi}{(\mu^2+1)^{\frac{3}{2}}(\tan^2\phi - \mu^2)}.$$

The values of ρ calculated for diamond from this formula are given in the following table.

ϕ...	60°	61°	62°	63°	64°	65°	66° 30′	67°	67° 30′
$-\rho$	·041	·046	·053	·062	·077	·103	·203	·288	·436
ϕ...	68°	68° 30′	69°	70°	71°	72°	73°	74°	75°
$-\rho$	·624	·754	·828	·897	·928	·946	·957	·965	·971

Also we have very approximately

$$x = -\frac{\tan(\phi - \phi')}{\tan(\phi + \phi')} \text{ (the } r \text{ of Fresnel's formula),}$$

and
$$y = \frac{2(\mu^2 - 1)\epsilon_1}{(\mu^2+1)^{\frac{3}{2}}[\cos\phi + \cos\phi'/\mu]}.$$

The denominator in the formula for y will vary very little throughout a considerable range on each side of Brewster's angle, so that throughout this range y will be very nearly a constant. The following are the values of x and y for diamond.

ϕ	x	y	ϕ	x	y
60°	− ·1314	·0168	68°	·0071	·0173
61°	− ·1162	·0169	69°	·0287	·0173
62°	− ·1016	·0170	70°	·0514	·0173
63°	− ·0858	·0170	71°	·0754	·0172
64°	− ·0690	·0171	72°	·1007	·0172
65°	− ·0513	·0172	73°	·1247	·0170
66°	− ·0328	·0173	74°	·1525	·0170
67°	− ·0136	·0173			

Since y is practically constant throughout this range, if we draw a graph to represent the amplitude and phase of the displacement in the reflected light, we get very approximately a straight line parallel to the axis of x, and all the points of departure from Fresnel's formulae due to the layer are indicated, as regards both amplitude and phase, by the slight shifting of a straight line from the axis of x to a parallel position. This is illustrated in Fig. 22, where OP represents the amplitude of the displacement in the reflected wave, and the angle POx represents the change of phase produced by reflection.

The formulae already obtained enable us to estimate the influence of the surface layer of transition on the reflecting power of the substance. This power is measured, as we have seen in the

last chapter, by $(R^2 + R'^2)/2$, and as the formulae show that R^2 and R'^2 involve d_1^2, but not d_1, it follows that if the layer be so thin

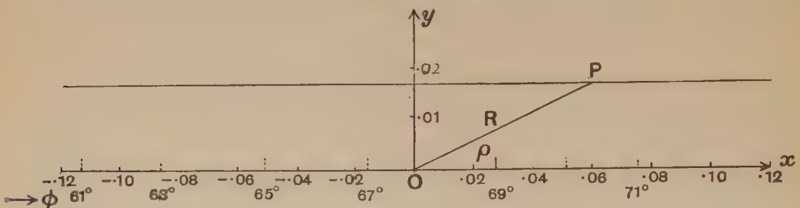

Fig. 22.

that the square of d_1 can be neglected, it has no influence on the reflecting power. If, then, we are considering experiments such as those referred to on p. 53, where there is a decided departure from Fresnel's formulae, it will be necessary to include terms containing d_1^2 in our approximations. This has been done above, and the formulae for R and R' there obtained may be written thus:

$$R = \frac{\mu\cos\phi - \cos\phi'}{\mu\cos\phi + \cos\phi'}\left[1 + \frac{2\mu^3 d_1^2\cos\phi\cos\phi'}{(\mu^2\cos^2\phi - \cos^2\phi')^2}\left\{\begin{array}{l} E^2\mu^2\cos^2\phi\cos^2\phi' + (1 - F/\mu^2 . \sin^2\phi)^2 \\ \quad - E(1 - F/\mu^2 . \sin^2\phi)(\mu^2\cos^2\phi + \cos^2\phi') \\ \quad + (\mu^2\cos^2\phi - \cos^2\phi')(E - 2H + (J - L)/\mu^2 . \sin^2\phi) \end{array}\right\}\right]$$

$$= \frac{\mu\cos\phi - \cos\phi'}{\mu\cos\phi + \cos\phi'}\left[1 + \frac{2\cos\phi\cos\phi'}{(\mu^2\cos^2\phi - \cos^2\phi')^2}(a + b\sin^2\phi + c\sin^4\phi)\right],$$

where

$$a = \mu^3 d_1^2 [E^2\mu^2 + 1 - 2\{E + (\mu^2 - 1) H\}],$$

$$b = \mu^3 d_1^2 [-E^2(\mu^2 + 1) + 2(E - F)/\mu^2 + EF(\mu^2 + 1)/\mu^2 + (J - L)(1 - 1/\mu^2) + 2H(\mu^2 - 1/\mu^2)],$$

$$c = \mu^3 d_1^2 [E^2 + F^2/\mu^4 - EF(1 + 1/\mu^4) - (J - L)(1 - 1/\mu^4)],$$

and

$$R' = \frac{\mu\cos\phi' - \cos\phi}{\mu\cos\phi' + \cos\phi}\left[1 + \frac{2\mu d_1^2\cos\phi\cos\phi'}{(\mu^2\cos^2\phi' - \cos^2\phi)^2}\left\{\begin{array}{l} \mu^2\cos^2\phi\cos^2\phi' + (E\mu^2 - \sin^2\phi)^2 \\ \quad + \mu^2(E - 2H)(\mu^2\cos^2\phi' - \cos^2\phi) \\ \quad - (E\mu^2 - \sin^2\phi)(\mu^2\cos^2\phi' + \cos^2\phi) \end{array}\right\}\right]$$

$$= \frac{\mu\cos\phi' - \cos\phi}{\mu\cos\phi' + \cos\phi}\left[1 + \frac{2\alpha\cos\phi\cos\phi'}{(\mu^2\cos^2\phi' - \cos^2\phi)^2}\right].$$

Thus, as far as the influence of the layer is concerned, the intensity of the reflected beam can be expressed in terms of three constants a, b, c, whose values depend on the thickness of the layer and the law of distribution of the refractive index within it.

Putting $R = R_1 + \rho_1$ and $R' = R_1' + \rho_1'$, where R_1 and R_1' have the values given by Fresnel's formulae, we see that the addition to the reflecting power due to the layer is $R_1\rho_1 + R_1'\rho_1'$. This addition is obtained from the above formulae in the form

$$2 \cos\phi \cos\phi' [aA + B\sin^2\phi\,(b + c\sin^2\phi)],$$

where $$A = 1/(\mu\cos\phi + \cos\phi')^4 + 1/(\mu\cos\phi' + \cos\phi)^4,$$

and $$B = 1/(\mu\cos\phi + \cos\phi')^4.$$

At normal incidence this quantity is positive or negative, according as a is positive or negative, so that the layer increases or diminishes the reflecting power at normal incidence, according as H is less or greater than

$$(E^2\mu^2 - 2E + 1)/2(\mu^2 - 1).$$

The values of the constants a, b, c could be determined from the reflecting powers at three different angles of incidence. Their values could also be calculated in terms of d_1 by means of the above formulae, for any given law representing the refractive index at different points within the layer. Thus in considering Conroy's experiments on reflection from glass we shall suppose that the refractive index within the layer is given by the formula

$$1/\mu'^2 = \sqrt{1 + px_1},$$

and take $p = 1/\mu^4 - 1$ so as to preserve continuity when $x_1 = 1$. The value of μ found by experiment was 1·5145, and taking $d_1 = 0{\cdot}481$ we get

$$a = -{\cdot}041,\quad b = {\cdot}06642,\quad c = -{\cdot}08079.$$

The following table gives the intensity of the reflected light, for different angles of incidence, calculated from the above formulae, with these values of the constants, and compares them with the results of Conroy's experiments.

Fig. 23 represents these results graphically.

ϕ...	0°	10°	20°	30°	40°	50°	60°	65°	70°
Intensity (Theory)	·0378	·0378	·0380	·0392	·0439	·0537	·0831	·1128	·1612
(Exp.)	·0378	·0378	·0377	·0392	·0437	·0553	·0854	·1116	·1604
Difference	0	0	+ ·0003	0	+ ·0002	− ·0016	− ·0023	+ ·0012	+ ·0008

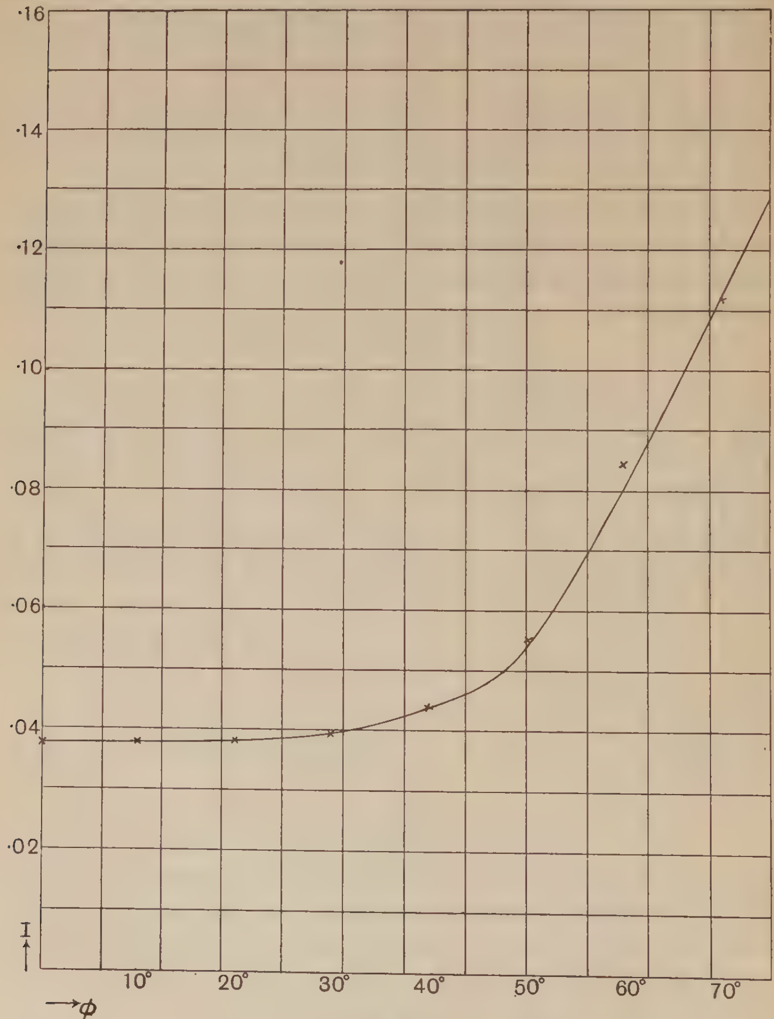

Fig. 23.

We have seen that one effect of a surface layer of transition is that there is no longer any angle of incidence, at which the component of the displacement at right angles to the plane of incidence is absent from the reflected beam. Thus the reflected light can never be plane polarised. However, as the coefficient of ellipticity is always very small, the departure from plane polarisation will be slight, and it may be convenient for some purposes to speak of the polarising angle as the angle of incidence, for which the component of the displacement at right angles to the plane of incidence is least in the reflected light. The position of this angle is easily obtained from our formulae. We have

$$r=\frac{(\mu\cos\phi-\cos\phi')+i\mu d_1\,(E\mu\cos\phi\cos\phi'-1+F/\mu^2\,.\sin^2\phi)}{(\mu\cos\phi+\cos\phi')+i\mu d_1\,(E\mu\cos\phi\cos\phi'+1-F/\mu^2\,.\sin^2\phi)}.$$

At Brewster's angle $\mu\cos\phi-\cos\phi'=0$, so that in its neighbourhood we have $R^2=x^2+y^2$, where

$$x=\frac{\mu\cos\phi-\cos\phi'}{\mu\cos\phi+\cos\phi'},$$

and
$$y=\mu d_1\frac{(E\mu\cos\phi\cos\phi'-1+F/\mu^2\,.\sin^2\phi)}{\mu\cos\phi+\cos\phi'}.$$

The polarising angle, as here defined, occurs when R is a minimum, i.e. when

$$x\,dx/d\phi+y\,dy/d\phi=0.$$

Since y, being proportional to d_1, is of the first order of small quantities, $y\,dy/d\phi$ is of the second order. Hence if we neglect d_1^2 we have $x=0$, i.e. $\mu\cos\phi-\cos\phi'=0$, which corresponds to Brewster's angle $\alpha=\tan^{-1}\mu$. To this order, then, the layer does not alter the position of the polarising angle. Proceeding to a higher order of approximation R will be a minimum when $\phi=\alpha-\psi$, where ψ is a small angle to be determined. Since

$$f(\alpha-\psi)=f(\alpha)-\psi f'(\alpha)$$

approximately, the equation to determine ψ may be written

$$(x-\psi\,dx/d\phi)(dx/d\phi-\psi\,d^2x/d\phi^2)$$
$$+(y-\psi\,dy/d\phi)(dy/d\phi-\psi\,d^2y/d\phi^2)=0,$$

where ϕ is to be put equal to α after differentiating. Since $x = 0$ when $\phi = \alpha$, this equation gives

$$\psi = \frac{y dy/d\phi}{(dx/d\phi)^2} = \frac{\mu^3 d_1^2}{2(\mu^2+1)(\mu^4-1)^2}[1+\mu^2-F-E\mu^2]$$
$$[(1+\mu^2)(1+\mu^4)-F(1+4\mu^2+\mu^4)+E\mu^2(1+\mu^4)],$$

where ψ is expressed in circular measure, and represents the diminution of the polarising angle produced by the layer. With the law giving the refractive index within the layer that was discussed above, the diminution amounts to $10'\,12''$, and the polarising angle becomes $56°\,23'\,38''$. The mean of the results obtained experimentally by Conroy was $56°\,23'\,30''$, so that here, as elsewhere, there is a very close agreement between theory and observation.

The methods, hitherto used in discussing the influence of the layer of transition, may be applied equally well when dealing with the problem of total reflection. Thus for light polarised at right angles to the plane of incidence we have $\xi = 0 = \eta$. In the first and second media ζ is given by the formulae of p. 56, and in the layer by the formula $\zeta = ue^{ip(t-\nu y/c)}$, when u has the same value as on p. 65. Neglecting squares of d_1 the boundary conditions give

$$A = 1 - r, \qquad B\mu^2 = -id_1\kappa(1+r),$$
$$s = A + BE, \qquad -in\mu s = B/d_1 + d_1 A(\nu^2 F - 1),$$

where $\kappa = \mu\cos\phi$.

Putting $$n = -i\sqrt{\sin^2\phi - 1/\mu^2} = -ip,$$
and eliminating A and B, we get

$$s + r(1 + id_1\kappa E/\mu^2) = 1 - id_1\kappa E/\mu^2,$$

and $$p_1\mu s - r[i\kappa/\mu^2 - d_1(1 - F\nu^2)] = i\kappa/\mu^2 + d_1(1 - F\nu^2).$$

From these r and s can be determined. For reflection we are interested only in r, which is given by the formula

$$r = \frac{p_1 - d_1(1-F\nu^2)/\mu - i(\kappa/\mu^3 + d_1 p_1\kappa E/\mu^2)}{p_1 - d_1(1-F\nu^2)/\mu + i(\kappa/\mu^3 + d_1 p_1\kappa E/\mu^2)} = e^{i(2\alpha'-\pi)},$$

where

$$\tan\alpha' = \frac{p_1 - d_1(1-F\nu^2)/\mu}{\kappa/\mu^3 + d_1 p_1 \kappa E/\mu^2} = \frac{p_1\mu^3}{\kappa}\left[1 - \frac{d_1}{p_1\mu}\{1 - E - (F-E)\nu^2\}\right]$$
$$= \tan\alpha - x_1,$$

where $$\tan\alpha = \frac{p_1\mu^3}{\kappa} = \frac{\mu^2\sqrt{\sin^2\phi - 1/\mu^2}}{\cos\phi},$$

so that α is the same angle as that used in dealing with the case of an abrupt transition, and

$$x_1 = \mu d_1\{1 - E - (F - E)\nu^2\}/\cos\phi.$$

Similarly, for light polarised parallel to the plane of incidence, we have ζ' in the first and second media given by the formulae of p. 57, and in the layer by $\zeta' = ue^{ip(t-\nu y/c)}$, where u has the same meaning as on p. 69. The boundary conditions give

$$A = 1 + r', \qquad B = -id_1\kappa(1 - r'),$$

$$s' = A + B, \qquad -in\mu s' = B/d_1 + d_1 A(\nu^2 - E).$$

Eliminating A and B, we get

$$s' - r'(1 + id_1\kappa) = 1 - id_1\kappa,$$

and $$p_1\mu s' + r'\{i\kappa - d_1(E - \nu^2)\} = i\kappa + d_1(E - \nu^2),$$

whence $$r' = \frac{-p_1\mu + d_1(E - \nu^2) + i\kappa(1 + d_1 p_1\mu)}{p_1\mu - d_1(E - \nu^2) + i\kappa(1 + d_1 p_1\mu)} = e^{2i\alpha''},$$

$$\tan\alpha'' = \frac{p_1\mu - d_1(E - \nu^2)}{\kappa(1 + d_1 p_1\mu)} = \frac{p_1\mu}{\kappa}\left[1 + \frac{d_1}{p_1\mu}(1 - E)\right]$$

$$= \frac{1}{\mu^2}(\tan\alpha + x_2), \quad \text{where} \quad x_2 = \mu d_1(1 - E)/\cos\phi.$$

The difference of phase between the components of the displacement parallel and perpendicular to the plane of incidence is

$$\Delta' = \pi - 2(\alpha' - \alpha''),$$

so that Δ' is given by the formula

$$\cot\frac{\Delta'}{2} = \tan(\alpha' - \alpha'') = \frac{\tan\alpha(1 - 1/\mu^2) - (x_1 + x_2/\mu^2)}{1 + \tan^2\alpha/\mu^2 + (x_2 - x_1)\tan\alpha/\mu^2}$$

$$= \frac{(\mu^2 - 1)p_1\sec\phi - (x_1 + x_2/\mu^2)}{(\mu^2 - 1)\tan^2\phi + (x_2 - x_1)p_1\sec\phi}$$

$$= \frac{p_1}{\sin\phi\tan\phi} - \frac{\mu d_1}{(\mu^2 - 1)\sin\phi\tan\phi}[1 - F + (1 - E)/\mu^2]$$

$$= \frac{p_1 - \kappa_1}{\sin\phi\tan\phi},$$

where $$\kappa_1 = \frac{\mu d_1}{\mu^2 - 1}\left[1 + \frac{1}{\mu^2} - F - \frac{E}{\mu^2}\right]$$

$$= \frac{\mu d_1}{\mu^2 - 1}\int_0^1 \left(1 - \frac{1}{\mu'^2}\right)\left(1 - \frac{\mu'^2}{\mu^2}\right) dx_1 = \pm \frac{2\epsilon_1}{\mu\sqrt{1+\mu^2}}.$$

Here ϵ_1 is the coefficient of ellipticity as formerly defined, and κ_1 is positive if μ' is less than μ.

The following table gives the values of Δ' calculated for different angles of incidence from the above formula, and compares them with the results of experiment.

The refractive index is $\mu = 1{\cdot}619$, the critical angle 38° 9′, and the coefficient of ellipticity $\epsilon_1 = {\cdot}0250$.

ϕ...	38° 13′	39° 58′	41° 59′	44° 2′	46° 4′	47° 54′	49° 58′	51° 57′	53° 58′	55° 57′
Δ' (Theory)	·975	·813	·758	·728	·716	·712	·712	·719	·727	·736
Δ' (Exp.)	·978	·813	·754	·728	·718	·712	·715	·719	·726	·730
Difference	− ·003	0	+ ·004	0	− ·002	0	− ·003	0	+ ·001	+ ·006

These results are shown in Fig. 24.

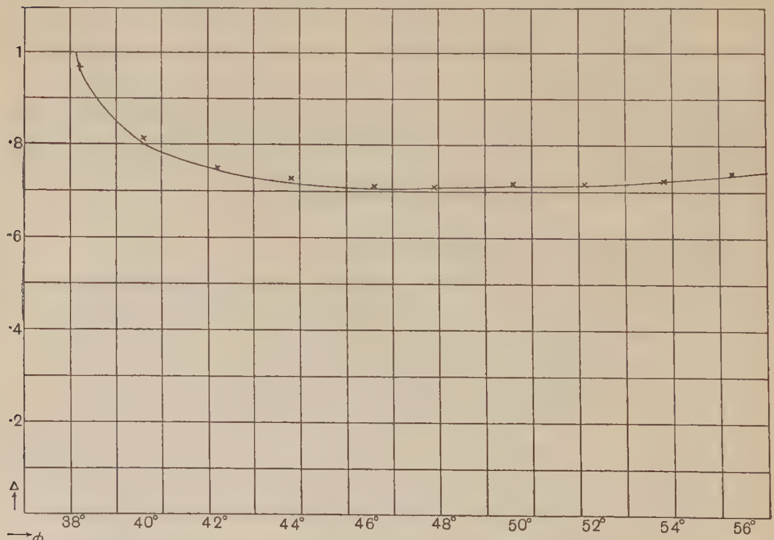

Fig. 24.

Having seen that the deductions from theory agree thoroughly with the results of careful experiment, wherever a comparison is possible, we feel that we are on solid ground when we investigate the details of the polarisation of the reflected light, even although our results cannot be put directly to the test of experiment. Let us suppose, then, that the incident light is polarised in a plane making an angle θ with the plane of incidence, and that the displacement is of unit amplitude. This displacement may be resolved into two components

$$\xi_1 = \cos\theta \cos pt, \quad \text{and} \quad \eta_1 = \sin\theta \cos pt,$$

parallel and perpendicular respectively to the plane of incidence. After reflection the components will be given by the formulae

$$\xi_1 = R' \cos(pt + \rho'), \quad \text{and} \quad \eta_1 = R\cos(pt + \rho),$$

where R, R', ρ, ρ' have the values assigned to them earlier in this chapter. We have seen (p. 52) that ξ_1 lags in phase behind η_1 by an amount $\Delta = \rho - \rho'$, so that in cases of positive reflection for which Δ is negative, the component of the displacement perpendicular to the plane of incidence lags behind the other component. On eliminating t from the equations for ξ_1 and η_1, we get

$$\xi_1^2/R'^2\cos^2\theta + \eta_1^2/R^2\sin^2\theta - 2\xi_1\eta_1\cos\Delta/RR'\sin\theta\cos\theta = \sin^2\Delta.$$

Thus the reflected beam is, in general, elliptically polarised, the elements of the elliptical orbit being determined by means of the above equation.

Putting $\tan\alpha = \epsilon\tan\theta$, where $\epsilon = R/R'$, this equation becomes

$$\xi_1^2\tan^2\alpha + \eta_1^2 - 2\xi_1\eta_1\tan\alpha \, . \cos\Delta = R^2\sin^2\Delta\sin^2\theta.$$

The sense in which this orbit is described is determined by considering the signs of ξ_1, η_1, $\dfrac{\partial\xi_1}{\partial t}$, and $\dfrac{\partial\eta_1}{\partial t}$. We shall take the case of positive reflection and look along the ray, whether incident or reflected, in the direction of the propagation of the light. If θ be positive, we have to turn from the plane of incidence counter-clockwise through an angle θ to get to the direction of the displacement in the incident wave, and in this case the orbital motion in the reflected wave is clockwise.

The orbit is represented by the ellipse in Fig. 25, in which OE and OF represent the amplitudes of the displacements parallel and

perpendicular to the plane of incidence, so that $OE = R' \cos\theta$ and

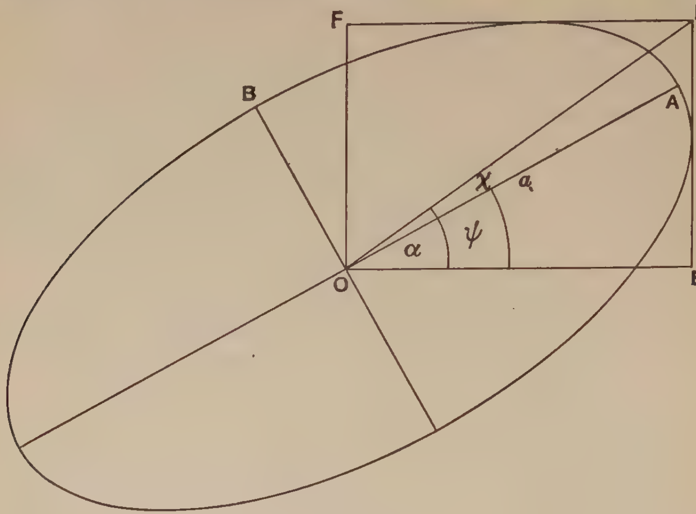

Fig. 25.

$OF = R \sin\theta$. Let OA and OB be the semi-axes (a and b) of the ellipse, $AOE = \psi$, $HOA = \chi$, $HOE = \alpha$, so that

$$\tan\alpha = OF/OE = \epsilon \tan\theta.$$

The azimuth ψ of the major axis of the ellipse is given by the equation

$$\tan 2\psi = \frac{2h'}{a' - b'} = \frac{2\tan\alpha \, . \cos\Delta}{1 - \tan^2\alpha} = \tan 2\alpha \, . \cos\Delta,$$

and the eccentricity e by the equation

$$e^4/(2 - e^2)^2 = 1 - 4(a'b' - h'^2)/(a' + b')^2 = 1 - \sin^2 2\alpha \, . \sin^2\Delta.$$

Putting $\sin 2\gamma = \sin 2\alpha \, . \sin\Delta$, we have $e = \sec\gamma \sqrt{\cos 2\gamma}$,

$$a = R \sin\theta \cos\gamma \operatorname{cosec}\alpha, \quad b = a \tan\gamma.$$

We shall consider first the variation of ψ for different incidences (ϕ) and azimuths (θ). At normal incidence $\epsilon = 1$ and $\Delta = 0$, so that $\psi = \alpha = \theta$. As the angle of incidence increases ψ diminishes, and as a rule continues to do so until it vanishes at the Principal Incidence, where $\cos\Delta = 0$. After this ψ becomes negative, and continues to diminish algebraically until the incidence is grazing, when $\psi = -\theta$. For some azimuths, however, ψ does not vanish at the Principal Incidence, but is 90° there. At this

incidence we have $\epsilon = \epsilon_1 = \cot \Omega$, so that if θ be greater than Ω, we have $\tan \alpha$ greater than unity, and therefore 2α greater than $90°$. Hence, from the formula $\tan 2\psi = \tan 2\alpha \,.\, \cos \Delta$, we see that just below the Principal Incidence $\tan 2\psi$ is negative, and ψ is rather less than $90°$, while just beyond the Principal Incidence $\tan 2\psi$ is positive, and ψ is rather greater than $-90°$. It appears, then, that if θ be greater than Ω the curve representing ψ in the neighbourhood of the Principal Incidence takes the form represented in Fig. 26.

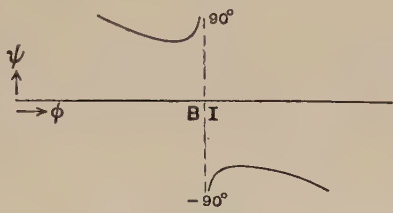

Fig. 26.

Thus at the Principal Incidence in passing through the azimuth $\theta = \Omega$, the angle ψ changes suddenly from $0°$ to $90°$. This means, of course, that the longer axis of the ellipse is the one at right angles to the plane of incidence, instead of being in that plane as before. There is, however, no real discontinuity at $\theta = \Omega$, for at that azimuth the ellipse is a circle, as we shall find immediately.

If there were no layer the reflected light would be plane polarised in an azimuth θ' given by $\tan \theta' = \pm \epsilon \tan \theta$, the upper or lower sign being taken, according as the angle of incidence is less or greater than the Principal Incidence. Except near the Principal Incidence the value of ϵ is the same whether there is a layer or not, so that $\tan \theta' = \pm \tan \alpha$, or $\theta' = \pm \alpha$. Hence OH in Fig. 25 represents the direction of the displacement when the transition is abrupt, and the angle $\chi = \alpha - \psi$ represents the turning towards the plane of incidence in going from the azimuth of plane polarisation, when there is no layer, to the major axis of the elliptical orbit when the layer is present. It is obvious from the figure, or the formulae, that χ is positive if $\alpha < 45°$, and negative if $\alpha > 45°$. We have $\alpha > 45°$ when $\epsilon \tan \theta > 1$, and this will be the case at low and high incidences, when ϵ is nearly unity, for

azimuths θ greater than 45°, and for all incidences with azimuths greater than Ω. In the neighbourhood of the Principal Incidence ϵ is not the same for an abrupt transition as when there is a transition layer, so that θ' is no longer equal to $\pm\alpha$. When there is no layer we have $\epsilon = 0$ at the Principal Incidence, and $\theta' = 0$, so that $\chi = \theta' \sim \psi = 0$ if $\theta < \Omega$, and $\chi = 90°$ if θ is equal to or greater than Ω.

The table below gives the values of χ for different incidences and azimuths for reflection from diamond and realgar.

These substances have nearly equal refractive indices, but the coefficients of ellipticity found experimentally by Jaurin were very different, so that a comparison of the two cases indicates the effect of varying the surface layer. For diamond we have $\mu = 2{\cdot}434$, $\epsilon_1 = {\cdot}0241$, so that Brewster's angle is 67° 40′, and $\Omega = 88°\ 37'$. For realgar we have $\mu = 2{\cdot}420$, $\epsilon_1 = {\cdot}1120$, so that Brewster's angle is 67° 33′, and $\Omega = 83°\ 37'$. In the table with the various azimuths (θ), the first row gives the values of χ for diamond and the second for realgar.

θ	$\phi = 10°$	30°	60°	65°	67° 30′	67° 40′	68°	70°	75°	80°
10°	0	0	0°1′	0°2′	0°0′	0°0′	0°0′	0°2′	0°1′	0°0′
	0	0	0 14	0 0	0 0	0 0	0 0	0 0	0 12	0 6
30°	0	0	0 2	0 7	0 1	0 0	0 1	0 8	0 2	0 1
	0	0	0 45	0 2	0 1	0 1	0 1	0 3	0 39	0 16
45°	0	0	0 6	0 11	0 2	0 0	0 1	0 14	0 4	0 1
	0	0	1 12	0 8	0 1	0 0	0 1	0 7	1 1	0 19
60°	0	0	0 5	0 19	0 2	0 0	0 3	0 23	0 5	0 1
	0	0	1 40	0 25	0 1	0 2	0 4	0 23	1 18	0 10
75°	0	0	0 2	0 35	0 6	0 0	0 5	1 5	0 1	0 1
	0	0	0 58	0 23	0 5	0 11	0 38	1 12	0 20	−0 13
85°	0	0	0 5	0 12	0 34	0 0	0 15	0 23	0 8	0 1
	0	0	−1 45	−25 17	−87 47	−84 26	−69 34	−24 48	−1 16	−0 13
89°	0	0	−0 1	−0 31	−58 26	−90 0	−39 36	−0 42	−0 1	0 0
	0	0	−0 33	−9 36	−85 50	−79 1	−51 42	−9 24	−0 22	−0 3

The march of the quantity b/a is easily followed by means of the relations $b/a = \tan\gamma$, and $\sin 2\gamma = \sin\Delta \,.\, \sin 2\alpha$. For a given incidence $\sin 2\gamma$ varies as $\sin 2\alpha$, so that γ is greatest when $\alpha = 45°$,

and its greatest value is $\Delta/2$ below the Principal Incidence, and $(\pi - \Delta)/2$ beyond that. Thus as the azimuth (θ) varies, b/a is a maximum when $\tan\theta = 1/\epsilon$, and its greatest value is $\tan\Delta/2$ below the Principal Incidence, and $\cot\Delta/2$ beyond that. At the Principal Incidence itself $\Delta = \pi/2$, so that the greatest value of b/a is 1, and this occurs when $\tan\theta = 1/\epsilon_1$, i.e. when $\theta = \Omega$. Thus at the Principal Incidence, when $\theta = \Omega$ the reflected light is circularly polarised. The following table gives the maxima for different incidences and their corresponding azimuths.

ϕ......	10°	30°	60°	65°	67° 30′	67° 40′	68°	70°	75°	80°
b/a (diamond)	·0006	·0057	·0515	·1521	·8016	1	·6822	·1727	·0515	·0233
θ (diamond)	45° 42′	51° 55′	78° 24′	85° 43′	88° 35′	88° 37′	88° 30′	85°	76° 47′	66° 43′
b/a (realgar)	·0026	·0262	·2469	·5441	·9879	·9702	·8889	·5403	·2132	·1051
θ (realgar)	45° 44′	51° 56′	78° 30′	82° 21′	83° 37′	83° 35′	83° 30′	82° 18′	76° 40′	66° 35′

Figs. 27 and 28 represent these results graphically, giving the

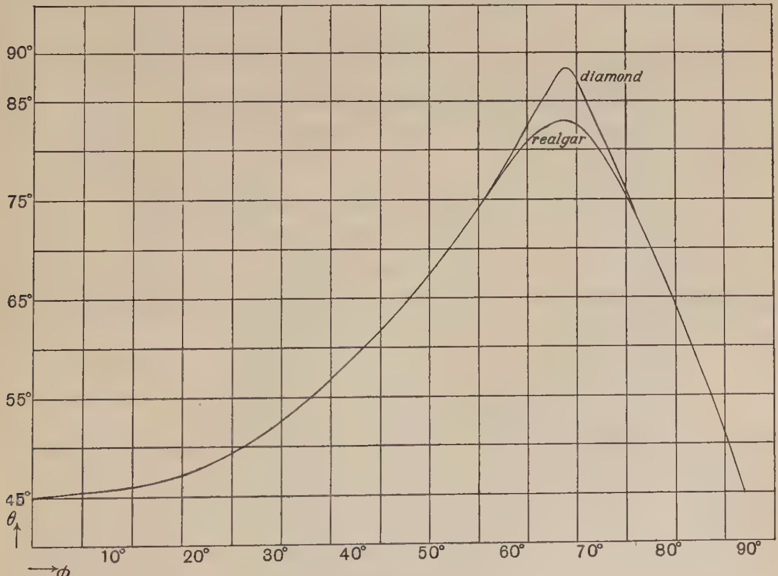

Fig. 27.

plane and elevation of the ridge of maxima. Fig. 27 shows the values of θ where the maxima occur, and Fig. 28 the values of these maxima.

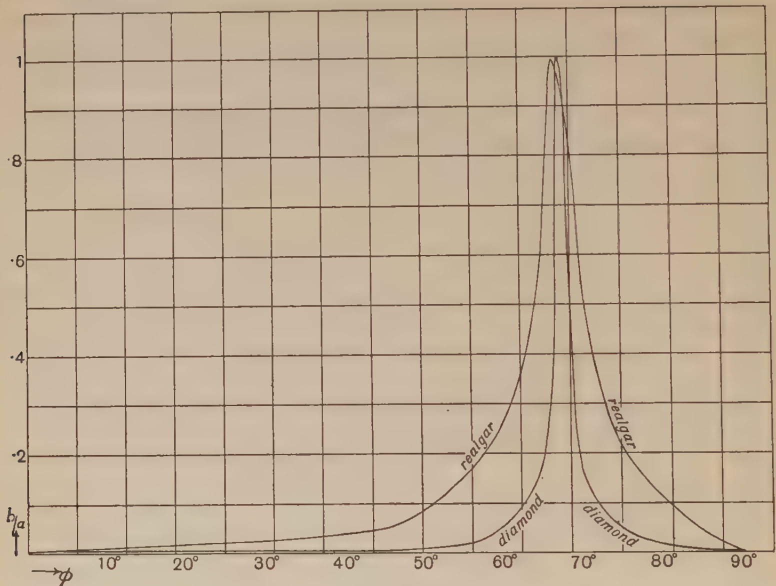

Fig. 28.

In seeking for the azimuth (θ) that makes b a maximum at any given angle of incidence, we take the equation

$$b^2 = a^2 \tan^2 \gamma = (R^2 \sin \Delta/2\epsilon) \sin 2\theta \tan \gamma = \kappa' \sin 2\theta \tan \gamma.$$

This gives

$$2b\,db/\kappa' = 2 \cos 2\theta \tan \gamma d\theta + \sin 2\theta \sec^2 \gamma d\gamma$$
$$= \sin 2\gamma \sec^2 \gamma d\theta \,[\cos 2\theta + \cos 2\alpha \sec 2\gamma].$$

Hence b is a maximum when $\cos 2\theta = -\cos 2\alpha \sec 2\gamma$. For brevity we shall put $\tan^2 \alpha = x$, $\sin^2 \Delta = s$, $\epsilon^2 = c$, and we then have to solve the equation

$$\frac{(1-x)(c+x)}{(1+x)(c-x)} = -\sqrt{1 - \frac{4sx}{(1+x)^2}}.$$

Squaring, and clearing of fractions, and noting that the process of squaring may introduce an irrelevant root, we get

$$x^2(1-c-s) + 2scx - c + c^2(1-s) = 0,$$

whence
$$x = -\frac{sc \pm (1-c)\sqrt{c(1-s)}}{1-c-s}.$$

To exclude the irrelevant root we must make the sign of

$$(1-x)(c+x)/(1+x)(c-x)$$

negative, and as $(c+x)/(1+x)$ is necessarily positive, this requires $(1-x)/(c-x)$ to be negative. Hence we get

$$(1-x)/(c-x) = \mp \cos\Delta/\epsilon,$$

taking the upper sign below the Principal Incidence and the lower sign beyond it. Thus we have

$$\tan^2\theta = (1 \pm \epsilon\cos\Delta)/\epsilon(\epsilon \pm \cos\Delta),$$

which gives real values of θ, whether the incidence is below or above the Principal Incidence. The maximum value of b corresponding to this azimuth is

$$R\sin\Delta/\sqrt{1 \pm 2\epsilon\cos\Delta + \epsilon^2}.$$

The following table gives the magnitude and position of the maximum value of b for various incidences.

ϕ......	0°	10°	30°	60°	65°	67° 30′
θ (diamond)	45°	45° 22′	48° 29′	65° 40′	75°	85° 33′
b (diamond)	0	·0002	·0023	·0112	·0143	·0170
θ (realgar)	45°	45° 22′	48° 30′	66° 19′	73° 46′	83° 17′
b (realgar)	0	·0011	·0107	·0521	·0714	·0786
ϕ......	67° 40′	68°	70°	75°	80°	90°
θ (diamond)	88° 37′	84° 14′	75° 34′	64° 11′	56° 45′	45°
b (diamond)	·0172	·0172	·0161	·0155	·0119	0
θ (realgar)	82° 48′	80° 51′	73° 40′	64° 40′	56° 47′	45°
b (realgar)	·0785	·0783	·0769	·0625	·0539	0

These results are represented in the figures below. Fig. 29 gives the values of the azimuths, and Fig. 30 the corresponding maxima.

A simple geometrical construction for $\tan^2\theta$, and for the corresponding maximum value of b, is easily obtained, see Fig. 31. Take

$$AB = 1,\quad AC = \epsilon,\quad BAC = \pi - \Delta \text{ or } \Delta,$$

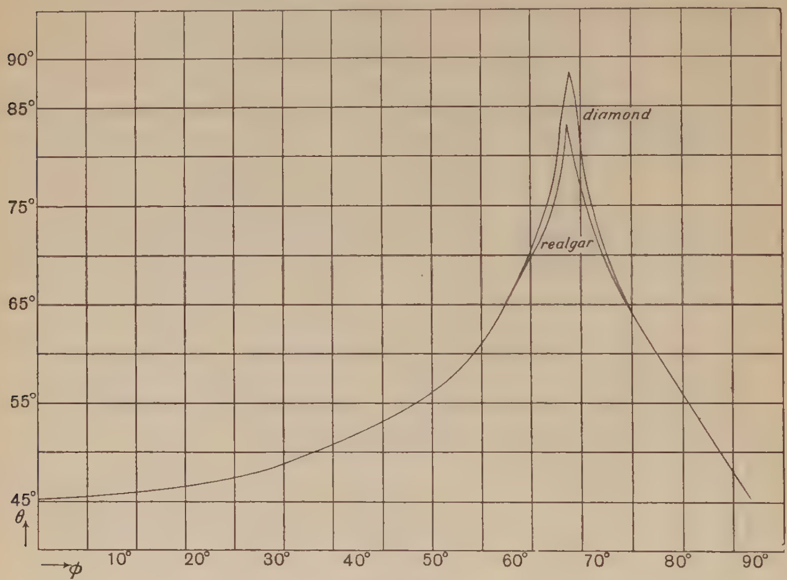

Fig. 29.

according as Δ is less or greater than $\pi/2$. Draw AD perpendicular to BC, and take P on AD such that $AP = R' \times AD$. Then $b = AP$, and $\tan^2\theta = BD/CD$. For

$$\sin\Delta/\sqrt{1 \pm 2\epsilon\cos\Delta + \epsilon^2} = BE/BC = \sin C,$$

and $R = \epsilon R' = R' \,.\, AC$. Hence

$$b = R\sin\Delta/\sqrt{1 \pm 2\epsilon\cos\Delta + \epsilon^2} = R' \,.\, AC\sin C = R' \,.\, AD.$$

Also $\qquad \tan^2\theta = (1 \pm \epsilon\cos\Delta)/\epsilon(\epsilon \pm \cos\Delta) = BF/AC \,.\, CE,$

and $\qquad BF = BF \,.\, AB = BC \,.\, BD,$

while $\qquad AC \,.\, CE = CD \,.\, CB;$

hence $\qquad \tan^2\theta = BD/CD.$

The changes in the magnitudes of θ and b are easily discussed by means of this construction. As Δ increases from 0 to $\pi/2$, BC/CD

increases from unity. It afterwards diminishes to unity. Thus θ increases from 45° to Ω, and then diminishes to 45°. Moreover, as

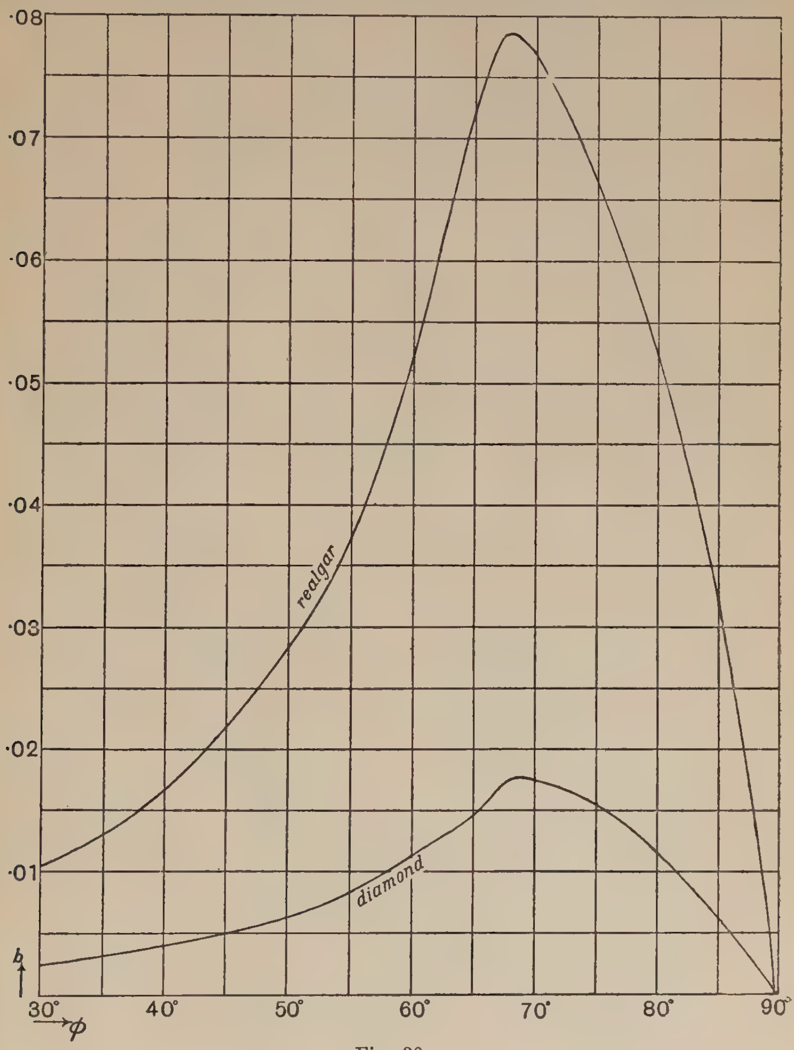

Fig. 30.

Δ increases from 0 to $\pi/2$, AD increases from 0 to $\epsilon_1/\sqrt{1+\epsilon_1^2}$, and R' increases, so that b increases in this range from 0 to $R_1/\sqrt{1+\epsilon_1^2}$, where R_1 is the value of R at the Principal Incidence. After this

AD diminishes to zero, while R' increases to unity, so that b diminishes to zero. In the immediate neighbourhood of the

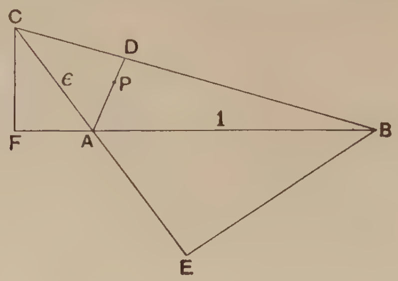

Fig. 31.

Principal Incidence we have $R \sin \Delta = R_1$, so that in this region $b = R_1/BC$. As the incidence increases beyond the Principal Incidence, BC increases so that b diminishes. Thus the maximum value of b is found at the Principal Incidence, and its value there is $R_1/\sqrt{1+\epsilon_1^2}$.

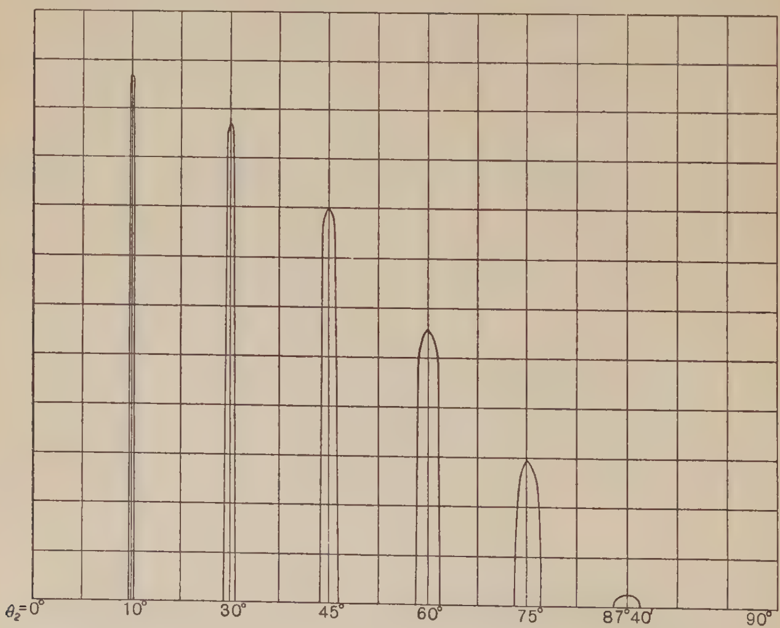

Fig. 32.

We have already observed that at the Principal Incidence the major axis of the elliptical orbit is in the plane of incidence for azimuths less than Ω, and at right angles to this for greater azimuths. The lengths of the axes at this incidence are most simply calculated by remembering that the amplitudes of the displacements parallel and perpendicular to the plane of incidence are always $R' \cos\theta$ and $R \sin\theta$. Hence the lengths of the semi-axes in these two directions are $R_1' \cos\theta$ and $R_1 \sin\theta$ respectively. Fig. 32 above and Fig. 33 below represent half the elliptical orbit for different azimuths at the Principal Incidence, Fig. 32 referring to diamond, and Fig. 33 to realgar.

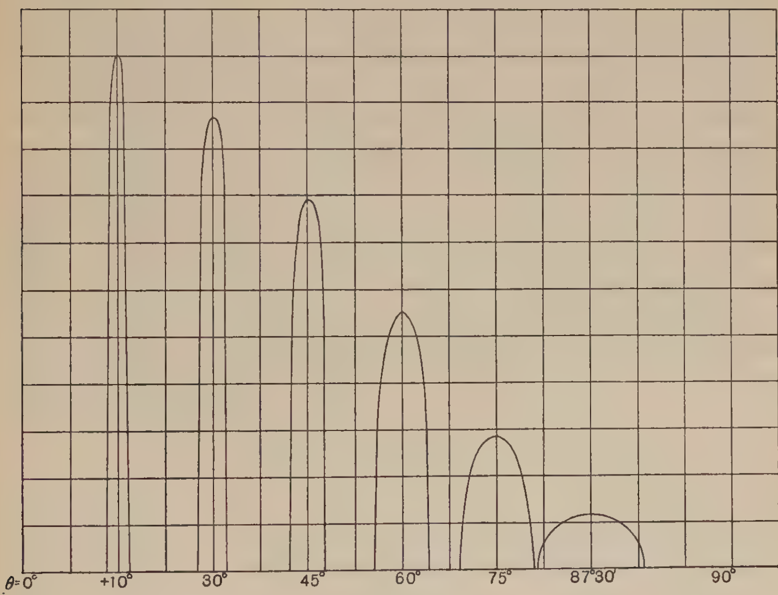

Fig. 33.

CHAPTER V.

TRANSPARENT ISOTROPIC PLATES. NEWTON'S RINGS.

THE results obtained in the last two chapters enable us to investigate the nature of the reflected and transmitted beam, when light is incident on a transparent isotropic film or plate, and so to explain, among other things, the colours of soap bubbles, of thin films of glass, or of a thin layer of oil on water. The phenomenon of this character that has been most carefully investigated experimentally is that of Newton's rings, and with this we shall deal more particularly; but we shall begin by obtaining a formula applicable to all such problems.

Suppose that we have a plate of thickness c_1, and of different refractive index from the media on each side of it. These media need not be of the same refractive index, e.g. we may have a film of soap between air and glass. A wave of unit amplitude repre-

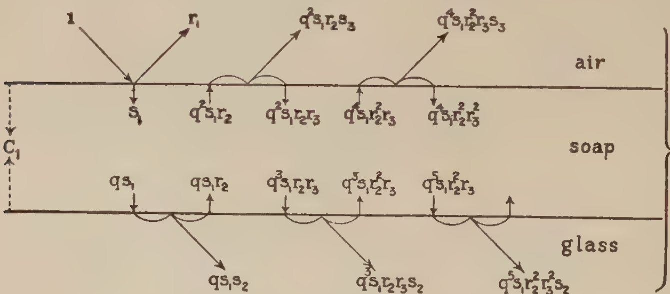

Fig. 34.

sented by $e^{i(pt-lx-my-nz)}$, will, at the passage from air to soap, give rise to a reflected wave $r_1e^{i(pt-lx-my-nz)}$ in the air, and a refracted wave $s_1e^{i(pt-lx-my-nz)}$ in the soap. Let the quantities corresponding

to r_1 and s_1 be r_2 and s_2 at the passage from soap to glass, and r_3 and s_3 at the passage from soap to air. Suppose, further, that a wave of unit amplitude, represented by $e^{i(pt-lx-my-nz)}$, becomes $qe^{i(pt-lx-my-nz)}$ after traversing a distance c_1 of the film or plate (of soap). Waves will be reflected to and fro between the air and glass, and those that return into the air will constitute a reflected beam represented (omitting the factor $e^{i(pt-lx-my-nz)}$) by

$$r_1 + q^2 s_1 r_2 s_3 (1 + q^2 r_2 r_3 + q^4 r_2^2 r_3^2 + \dots \text{ ad inf.})$$

$$= r_1 + \frac{q^2 s_1 r_2 s_3}{1 - q^2 r_2 r_3}, \text{ provided } |q^2 r_2 r_3| < 1.$$

Similarly the beam transmitted through the plate is represented by

$$q s_1 s_2 (1 + q^2 r_2 r_3 + q^4 r_2^2 r_3^2 + \dots) = \frac{q s_1 s_2}{1 - q^2 r_2 r_3}.$$

The values of the quantities r_1, r_2 etc., are given in Chap. III., for the ideal case of an abrupt transition, to which for the present we shall confine ourselves. From the formulae there obtained it appears that $r_1 + r_3 = 0$, and $s_1 s_3 = 1 - r_1^2 = 1 - r_3^2$, results that can also be obtained by noting that when the first and third media are the same, and the thickness of the film is indefinitely diminished, $r_3 = r_2$, $s_3 = s_2$, and $q = 1$, and that in this case all the light is transmitted, and none reflected. Making use of these relations we obtain for the reflected beam

$$\frac{r_1 + q^2 r_2}{1 + q^2 r_1 r_2}, \text{ and for the transmitted beam } \frac{q s_1 s_2}{1 + q^2 r_1 r_2}.$$

From these formulae the details both of the reflected and the transmitted system can be readily discussed. As, however, the media are transparent, there is no loss of light, so that the transmitted system is complementary to the reflected. Hence the investigation of the properties of one system involves those of the other, brightness in one case being replaced by darkness in the other.

If the incident light is of unit intensity, and is plane polarised in an azimuth θ with the plane of incidence, the intensity of the reflected beam is

$$I = \sin^2\theta \left| \frac{r_1 + q^2 r_2}{1 + q^2 r_1 r_2} \right|^2 + \cos^2\theta \left| \frac{r_1' + q^2 r_2'}{1 + q^2 r_1' r_2'} \right|^2,$$

where r and r' are given by Fresnel's formulae, and q^2 by the relation $q^2 = e^{-i\psi} = e^{-i\kappa\delta}$, where $\kappa = 2\pi/\lambda$, and $\delta = \mu'/\mu \,.\, c_1 \cos\phi'$. Here μ is the refractive index of the first medium, μ' that of the second, and λ the wave length in the first.

Thus we have

$$\left|\frac{r_1 + q^2 r_2}{1 + q^2 r_1 r_2}\right|^2 = \frac{r_1^2 + r_2^2 + 2r_1 r_2 \cos\psi}{1 + r_1^2 r_2^2 + 2r_1 r_2 \cos\psi}.$$

This is simply represented by the following geometrical construction. Take $OA = r_1$, $OB = r_2$, $AOB = \pi - \psi$, $OA' = 1$, and draw AB' parallel to $A'B$. Then $OB' = r_1 r_2$, and

$$\left|\frac{r_1 + q^2 r_2}{1 + q^2 r_1 r_2}\right|^2 = \left(\frac{AB}{A'B'}\right)^2.$$

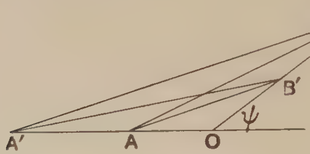

Fig. 35.

From this construction, or by means of the calculus, we see that the intensity I is stationary when $\psi = n\pi$, there being alternations of maxima and minima for different integral values of n. When $\psi = 2m\pi$ we have

$$I = \sin^2\theta \left(\frac{r_1 + r_2}{1 + r_1 r_2}\right)^2 + \cos^2\theta \left(\frac{r_1' + r_2'}{1 + r_1' r_2'}\right)^2,$$

and when $\psi = (2m+1)\pi$ we have

$$I = \sin^2\theta \left(\frac{r_1 - r_2}{1 - r_1 r_2}\right)^2 + \cos^2\theta \left(\frac{r_1' - r_2'}{1 - r_1' r_2'}\right)^2.$$

The difference between these two intensities is

$$D = \sin^2\theta \left[\frac{4r_1 r_2 (1 - r_1^2)(1 - r_2^2)}{(1 - r_1^2 r_2^2)^2}\right] + \cos^2\theta \left[\frac{4r_1' r_2' (1 - r_1'^2)(1 - r_2'^2)}{(1 - r_1'^2 r_2'^2)^2}\right].$$

Which of these stationary values of I is the bigger depends on the signs of r_1 and r_2. If $r_1 r_2$ be positive, so that r_1 and r_2 have the same sign, then the maxima occur when $\psi = 2m\pi$, while if $r_1 r_2$ be negative the maxima occur when $\psi = (2m+1)\pi$.

For a beam of parallel light for which ϕ, and therefore ϕ', is constant, ψ is constant, if the thickness of the film is everywhere the same. Such a film would therefore appear uniformly bright throughout its surface, the brightness depending on the thickness. If, however, as is nearly always the case in practice, the film is not uniformly thick, there will be alternations of brightness and darkness over the surface depending on the differences of thickness at various points. In the case of Newton's rings these alternations are distributed round a centre, where the thickness of the film is negligible. At the centre, where $c_1 = 0$, we have $\psi = 0$, so that the centre corresponds to $n = 0$ in the series of stationary values of I. Hence the central spot corresponds to the lowest of this series, and it is bright or dark, according as D is positive or negative.

If we are dealing with light polarised in the plane of incidence, then $\theta = 0$, and the sign of D is the same as that of $r_1' r_2'$, i.e. of $\dfrac{\sin(\phi - \phi')}{\sin(\phi + \phi')} \cdot \dfrac{\sin(\phi' - \phi'')}{\sin(\phi' + \phi'')}$. This is positive, if ϕ' is intermediate between ϕ and ϕ'', and negative otherwise. Hence the central spot is bright, if the refractive index of the film lies between those of the other media, but otherwise dark. In particular the central spot is dark, if the first and third media are the same.

For light polarised at right angles to the plane of incidence we have $\theta = 90°$, and the sign of D is the same as that of $r_1 r_2$, i.e. of $\dfrac{\tan(\phi - \phi')}{\tan(\phi + \phi')} \cdot \dfrac{\tan(\phi' - \phi'')}{\tan(\phi' + \phi'')}$. Let $\phi = I_1$ be the polarising angle at the first reflection, and $\phi = I_2$ that at the second. The first is always real; but the second may not be so. The greatest possible angle of incidence at the second surface is ϕ', where $\sin \phi' = \mu/\mu'$, so that $\tan \phi' = \mu/\sqrt{\mu'^2 - \mu^2}$. For the polarising angle at the second surface we have $\tan \phi' = \mu''/\mu'$, so that for I_2 to be real we must have $\mu'^2/\mu^2 < \mu'^2/\mu''^2 + 1$.

Let us suppose that μ' is intermediate between μ and μ''. Then, if ϕ lies between zero and I_1, $r_1 r_2$ is positive, and the rings are bright centred; if ϕ lies between I_1 and I_2, $r_1 r_2$ is negative, and the rings are dark centred, while if ϕ lies between I_2 and $90°$, $r_1 r_2$ is positive, and the rings are once more bright centred. If, on the other hand, μ' is not intermediate between μ and μ'', we must interchange the words bright and dark in this description of the

centres of the rings. The transition from bright to dark centred rings occurs when $D=0$, in which case there are no maxima and minima, for the rings disappear. The equation $D=0$ gives us

$$\tan^2\theta = -\frac{r_1'r_2'(1-r_1'^2)(1-r_2'^2)(1-r_1^2r_2^2)^2}{r_1r_2(1-r_1^2)(1-r_2^2)(1-r_1'^2r_2'^2)^2},$$

which determines the azimuth at which the rings disappear. From this we get $\theta = 90°$ when $r_1 = 0$ or $r_2 = 0$. The first corresponds to $\phi = I_1$, the polarising angle at the first surface, and the second to $\phi = I_2$ which, as has been seen, may not be real. In general, for θ to be real, we must have $r_1'r_2'$ of opposite sign to r_1r_2, and as these are always of the same sign if ϕ be less than I_1 or greater than I_2, the rings cannot disappear for incidences outside these limits.

The following table gives the values of θ calculated from the above formula for different incidences, and compares the results with the observations of Brewster. The media considered are air, soap, and diamond; $\mu' = 1{\cdot}475$, $I_1 = 55° 52'$, $\mu'' = 2{\cdot}44$.

ϕ......	55° 52′	60°	65°	67° 43′	70°	75°
θ (theory)	90°	74° 10′	67° 43′	64° 51′	62° 52′	59° 13′
θ (experiment)	90°	73°	68° 30′	66° 20′	63° 30′	59° 15′
Difference	0	1° 10′	− 0° 47′	− 1° 29′	− 0° 38′	− 0° 2′

The differences are well within the limits of error of Brewster's experiments. The rings disappear gradually, and not suddenly, so that it is difficult to determine with great precision when the disappearance takes place.

Whether the rings have bright or dark centres for any given incidence and azimuth is made evident by representing $90° - \theta$ graphically for different incidences.

We have

$$\tan^2(90° - \theta) = -\frac{r_1r_2(1-r_1^2)(1-r_2^2)(1-r_1'^2r_2'^2)^2}{r_1'r_2'(1-r_1'^2)(1-r_2'^2)(1-r_1^2r_2^2)^2},$$

so that there are two equal and opposite values of $90° - \theta$ for each value of ϕ, and $90° - \theta$ vanishes with r_1 and r_2, i.e. when $\phi = I_1$ or I_2. The figure will take different forms, according as I_2 is real

or not. Fig. 36 corresponds to the case where I_2 is real, and Fig. 36 *a* to that in which I_2 is not real.

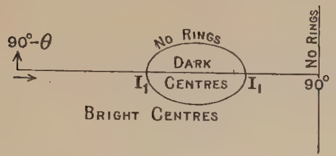

Fig. 36.

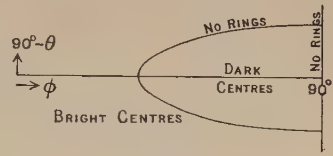

Fig. 36 *a*.

The regions are marked 'Bright centres' or 'Dark centres' for the case, where μ' is intermediate between μ and μ''. For any other case we must interchange the words Bright and Dark. For azimuths and incidences corresponding to points *on* the curves in these figures, the rings disappear.

The apparent brightness of the rings is proportional to the difference of the maxima and minima intensities, i.e. to D. The following table gives the values of D for different incidences and azimuths, for the case of air-soap-diamond already mentioned.

ϕ	$\theta=0°$	30°	45°	60°	75°	90°
0°	·172	·172	·172	·172	·172	·172
30°	·218	·195	·173	·151	·134	·128
55° 52′	·364	·273	·182	·091	·024	0
60°	·399	·291	·183	·076	− ·003	− ·032
65°	·441	·312	·182	·053	− ·041	− ·076
70°	·475	·325	·175	·025	− ·085	− ·125
75°	·496	·328	·160	− ·008	− ·131	− ·176
90°	0	0	0	0	0	0

Fig. 37 represents these results graphically.

The rings are bright centred or dark centred, according as D is positive or negative. The table, or the figure, shows clearly how the intensity varies, and how the change from bright to dark centres takes place. The shaded portion of the figure indicates the region in which D is negative, and the rings dark centred.

For azimuths less than 45° the distinctness of the rings increases with the angle of incidence as far as I_1, but it diminishes for greater

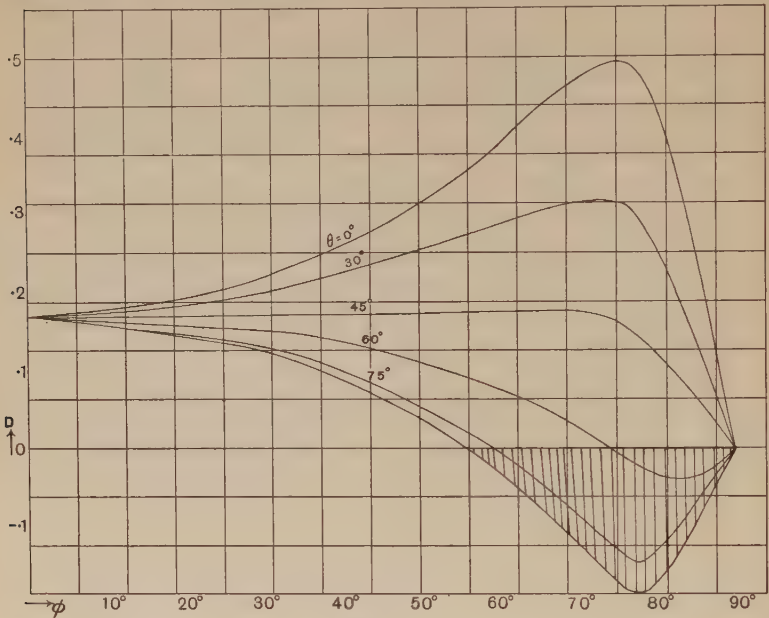

Fig. 37.

azimuths. When the angle of incidence lies between I_1 and 90°, the brightness increases for azimuths less than 45°, passes through a maximum, and then diminishes rapidly to zero; but for greater azimuths the brightness diminishes until the centres become dark, then passes to a minimum, and increases to zero at grazing incidence. Putting D in the form

$$D = D_1 \sin^2\theta + D_2 \cos^2\theta,$$

we see that for a given incidence the brightness continually diminishes as the azimuth increases. The brightest centred ring is found when $\theta = 0$, and the darkest centred one when $\theta = 90°$. From Fig. 37 we find the magnitude and position of the maxima and minima of brightness for the different azimuths represented are given as follows:—

θ	0°	30°	45°	60°	75°	90°
Maximum: D	·497	·328	·183	− ·045	− ·154	− ·190
ϕ	76°	75°	66°	82°	79°	78°

As an example of the same problem when I_2 is real, we shall take the case when the three media are air, water, and fluor-spar. For this we have

$$\mu' = 1{\cdot}336,\ \ \mu'' = 1{\cdot}437,\ \ I_1 = 53^\circ\, 11',\ \ I_2 = 78^\circ\, 4'.$$

The azimuths at which the rings disappear for different incidences are obtained from the formula of p. 98, which yields the following results.

ϕ	I_1	55°	60°	65°	70°	75°	I_2
θ	90°	89° 2′	87° 18′	86° 51′	87° 29′	88° 52′	90°

The brightness for different incidences and azimuths is set out in the following table of D, and in Fig. 38 below.

ϕ	$\theta=0°$	30°	45°	60°	75°	90°
0°	·018	·018	·018	·018	·018	·018
30°	·029	·025	·021	·018	·015	·014
55°	·060	·045	·031	·014	·003	− ·001
60°	·074	·055	·039	·016	·002	− ·004
65°	·085	·063	·045	·018	·001	− ·005
70°	·098	·073	·051	·021	·003	− ·004
75°	·109	·081	·055	·026	·005	− ·002
80°	·111	·083	·056	·029	·009	+ ·002
85°	·080	·061	·042	·023	·009	+ ·004
90°	0	0	0	0	0	0

All the noteworthy features of the phenomena are made evident by the figure. The rings pass from bright centred to

dark centred at I_1, and change again at I_2. Between I_1 and I_2 the intensity is always small, reaching a maximum about half-way

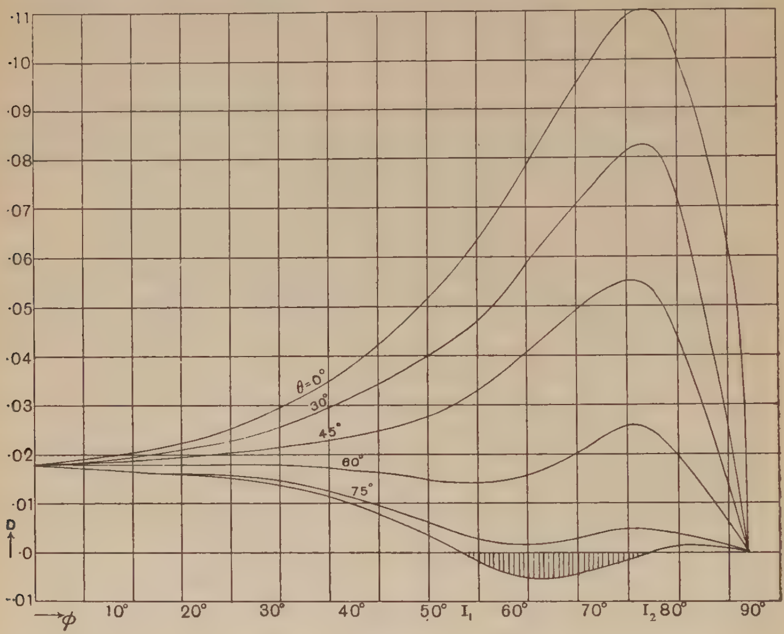

Fig. 38.

between I_1 and I_2. The azimuth required to produce these dark centred rings is very nearly 90°. When the white centred rings reappear, their intensity is nowhere large, reaching a maximum about midway between I_2 and 90°, though rather nearer the latter. The magnitudes and positions of the various maxima and minima obtained from the above figure are given in the following table. For values of θ greater than 45°, D has both a maximum and a minimum; the first entry in the table refers to the maxima and the second to the minima.

θ	0°	30°	45°	60°	75°	90°
Max. and min.: D	·112	·083	·057	·030 and ·014	·009 and ·001	·004 and – ·005
ϕ	79°	80°	80° 15′	81° and 55°	83° and 65°	85° and 67°

These various deductions from theory relative to the behaviour

of the rings for different incidences and azimuths are in complete harmony with the experimental results.

A specially simple and important application of the principles employed in this chapter is met with, when we are dealing with the case where the first and third media are the same. This happens in the ordinary arrangement for the production of Newton's rings, where a thin film of air separates two pieces of glass of the same kind. In this case we have to put

$$r_1 + r_2 = 0 = r_1' + r_2'$$

in the formulae of p. 96, whence we find that when $\psi = 2m\pi$, $I = 0$, so that the central spot and all the minima are not only dark, but absolutely black. The intensity of the bright rings, which occur when $\psi = (2m + 1)\pi$, is

$$I = \frac{4r^2}{(1 + r^2)^2}\sin^2\theta + \frac{4r'^2}{(1 + r'^2)^2}\cos^2\theta,$$

from which it appears that the rings cannot disappear, except when $\theta = 90°$ and $r = 0$, i.e. when the incident light is polarised at right angles to the plane of incidence, and the angle of incidence is the polarising angle. The following table gives the brightness of the rings for incidences less than the critical angle in the case of glass, for which $\mu = 1{\cdot}596$, the polarising angle being $32°\,1'$, and the critical angle $38°\,47'$. Fig. 39 represents these results graphically. The rings are black centred throughout.

ϕ	$\theta=0°$	30°	45°	60°	75°	90°
0°	·438	·438	·438	·438	·438	·438
10°	·459	·449	·439	·429	·422	·419
20°	·525	·479	·432	·386	·351	·339
30°	·690	·544	·398	·251	·144	·105
35°	·827	·680	·533	·386	·278	·239
38°	·958	·860	·763	·665	·593	·567
38° 47′	1	1	1	1	1	1

Having considered the brightness of the various rings, we must

now deal with their size. From the formulae above it appears

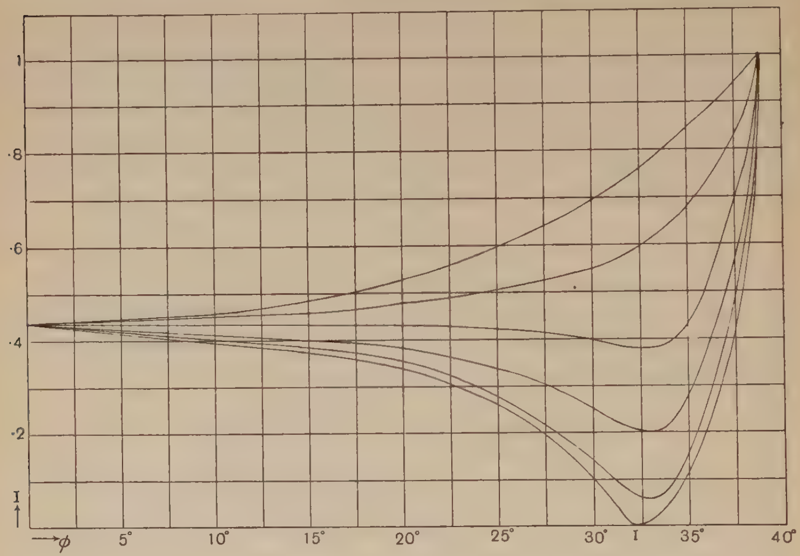

Fig. 39.

that I is stationary when $\psi = n\pi$, where n is any integer, and ψ is given by the equation

$$\psi = \frac{2\pi}{\lambda} \cdot \frac{\mu'}{\mu} c_1 \cos \phi'.$$

If r be the radius of a ring, κ_1 the curvature of the refractive surface, we have approximately $c_1 = \frac{1}{2}\kappa_1 r^2$. Hence we get

$$\rho = r\sqrt{(\kappa_1 \mu')/\lambda\mu} = \sqrt{n \sec \phi'},$$

the radius r being proportional to ρ. At the centre $n = 0$, and so $\rho = 0$, and by putting n equal to 1, 2, 3, 4 etc. we get the radii of the successive rings. It thus appears that the size of the rings is the same in whatever azimuth the incident light is polarised, and that it depends only on the angle of incidence within the film. For a given incidence the radii of the different rings are proportional to the square roots of the natural numbers, a law derived by Newton from his experiments. Newton also made observations on the dependence of the radii on the angle of incidence, the law according to theory being that the radius is proportional to the square root of sec ϕ'. This law has been more exactly verified

since Newton's time, but we give his results in the following table on account of their historical interest, and their accuracy considering the rough instruments of measurement at Newton's disposal.

ϕ'......	0°	10°	20°	30°	40°	50°
ρ (theory)	1	1·0077	1·032	1·075	1·142	1·247
ρ (Newton)	1	1·0077	1·033	1·075	1·140	1·250
ϕ'......	60°	65°	70°	75°	80°	85°
ρ (theory)	1·415	1·538	1·710	1·965	2·400	3·388
ρ (Newton)	1·4	1·525	1·690	1·925	2·286	2·9

The discrepancy between theory and observation is appreciable only for the larger angles of refraction, where a slight error in the estimate of the incidence makes a considerable difference in the radius. The values of ρ corresponding to the first bright ring for different angles of incidence are given in the next table, calculated for $\mu = 1{\cdot}596$.

ϕ	0°	10°	20°	30°	35°	38°	38° 15′	38° 30′	38° 46′	38° 47′
ρ	1	1·02	1·09	1·29	1·58	2·32	2·56	2·97	5·70	∞

The radii of the other rings, which are alternately dark and bright, are obtained by multiplying these values by the square roots of the natural numbers 2, 3, ... n. It appears from the table that the rings open out one after another on approaching the critical angle (38° 47′ in this case), and that they expand slowly at first, but afterwards very rapidly. The corresponding values of ρ for the first ring in the case dealt with above, where the three media are air, soap, and diamond, are as follows:—

ϕ	0°	30°	60°	65°	70°	75°	90°
ρ	1	1·03	1·11	1·13	1·14	1·15	1·17

Under such circumstances the opening out of the rings is a slow process throughout.

The most striking feature of the rings is their colour. That the rings should be coloured, when natural light is employed, is evident from the formula for the radius

$$r = \sqrt{n \sec \phi' . \mu\lambda/\mu' r},$$

which shows that the size of the ring depends on the wave length, and therefore differs for the different constituents of white light. To investigate accurately the dependence of the size of a ring on the colour of the light employed, we should need to know the relation between λ and μ, a relation that will be discussed in a later chapter on dispersion. It will be found, however, that the change in μ/μ' in going from one colour to another is usually small, so that we may, as a first approximation, regard μ'/μ as a constant, in which case the radius varies as the square root of the wave length. Thus the red rings are bigger than the violet, in accordance with the observation of Newton, who says, "I found the circles which the red light made to be manifestly bigger than those which were made by the blue and violet. And it was very pleasant to see them gradually swell or contract according as the colour of the light was changed." As a numerical example we shall calculate the relative size of the first and second rings corresponding to Fraunhofer's lines A, D, and H in the spectrum, A being in the extreme red, and H in the violet. We have approximately

$$r_a/r_d = \sqrt{\lambda_a/\lambda_d} = 1{\cdot}14 \quad \text{and} \quad r_h/r_d = \sqrt{\lambda_h/\lambda_d} = 0{\cdot}81,$$

whence we get the following results for $\mu = 1{\cdot}596$.

		$\phi=0°$	$\phi=20°$	$\phi=38°$
First bright ring	ρ_h	0·81	0·886	1·89
	ρ_d	1·00	1·09	2·32
	ρ_a	1·14	1·24	2·63
First dark ring	ρ_h	1·15	1·26	2·68
	ρ_d	1·41	1·55	3·29
	ρ_a	1·60	1·76	3·73

Having determined the radii of the rings, and the intensities at these rings (p. 103), we can draw curves to represent the

intensities at different distances from the centre, for any given incidence or azimuth. This is done in Figs. 40 and 41, which show the intensities for the three colours red, orange, and violet, corresponding to the lines A, D, H. Fig. 40 represents the case of normal incidence, and Fig. 41 that when the angle of incidence is 38°, and the light is plane polarised in the plane of incidence.

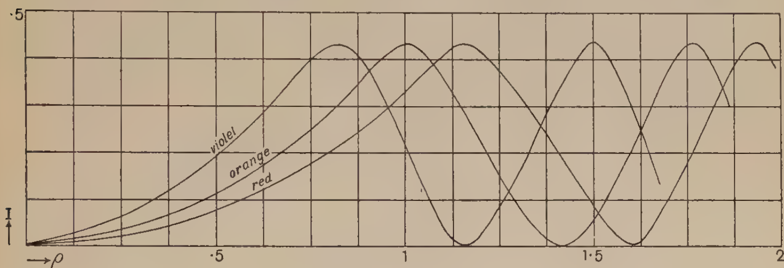

Fig. 40.

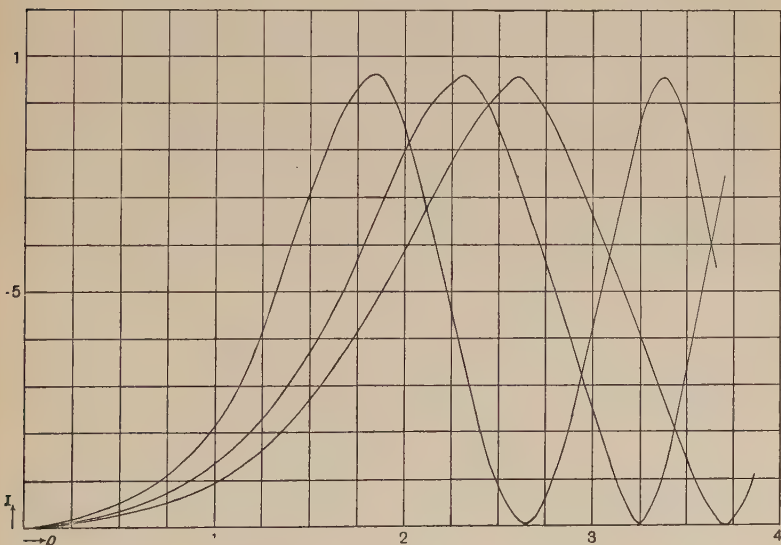

Fig. 41.

An inspection of these figures shows the general effect of the dependence of the size of the rings on the colour. Each bright ring changes in colour from violet on the inside to red on the outside, whereas in the dark rings the order of colours is reversed

It also appears that, as the order of the rings increases, the red of one ring tends to overlap the violet of the next. This diminishes the sharpness of the rings and makes the first more conspicuous than the others.

If the first medium be more highly refractive than the second we may, by employing a prism, make the angle of incidence on the film greater than the critical angle. Under these circumstances it is still true that the transmitted system is complementary to the reflected, and that the intensity of the latter is given in the most general case by the formula of p. 95, viz.

$$I = \sin^2\theta \left| \frac{r_1 + q^2 r_2}{1 + q^2 r_1 r_2} \right|^2 + \cos^2\theta \left| \frac{r_1' + q^2 r_2'}{1 + q^2 r_1' r_2'} \right|^2$$
$$= I_1 \sin^2\theta + I_2 \cos^2\theta,$$

but we must, of course, employ expressions for r, r', and q appropriate to the case of total internal reflection. When the first and third media are the same, which is the case that has been most carefully dealt with experimentally, we have

$$r_1 + r_2 = 0 = r_1' + r_2',$$

and
$$I = \sin^2\theta \left| \frac{r(1-q^2)}{1-q^2 r^2} \right|^2 + \cos^2\theta \left| \frac{r'(1-q^2)}{1-q^2 r'^2} \right|^2$$
$$= I_1 \sin^2\theta + I_2 \cos^2\theta.$$

The expressions for r, r', and q were obtained in Chap. III. and proved to be

$$r = e^{i(2\alpha-\pi)}, \quad r' = e^{i2\alpha'}, \quad q = \exp. \left[-\frac{2\pi c_1}{\lambda} \mu \sqrt{\sin^2\phi - 1/\mu^2} \right],$$

where λ is the wave length in air,

$$\tan\alpha' = \frac{\tan\alpha}{\mu^2} = \frac{\sqrt{\sin^2\phi - 1/\mu^2}}{\cos\phi}.$$

Also we have $2c_1/\lambda = \kappa_1 r^2/\lambda = \rho^2$, where r is the distance from the centre, and κ_1 the curvature of the refracting surface, so that ρ is proportional to the distance from the centre. Thus

$$q = \exp. [-\mu\pi\rho^2 \sqrt{\sin^2\phi - 1/\mu^2}], \quad I_1 = \frac{(1-q^2)^2}{(1-q^2)^2 + 4q^2 \sin^2 2\alpha},$$

and
$$I_2 = \frac{(1-q^2)^2}{(1-q^2)^2 + 4q^2 \sin^2 2\alpha'},$$

where I_1 and I_2 represent the intensities of the reflected beam for light polarised perpendicular and parallel respectively to the plane of incidence.

A consideration of the formula for I shows that there are no alternations of light and shade, as in the case of incidences below the critical angle, so that there are no longer any rings. At the centre $q=1$, so that $I=0$, and the central spot is perfectly black. As we recede from the centre q diminishes slowly at first but afterwards rapidly, so that I increases slowly, and then rapidly, giving the impression of a dark spot of sensible magnitude with its centre at the point of contact of the refracting surfaces. The table below, giving the numerical values of $B=1-I$ for different incidences when $\mu=1{\cdot}596$, shows more precisely the manner in which the blackness decreases from the centre.

If we wish to compare the intensity at a given distance from the centre in the two cases, when the light is polarised perpendicular and parallel to the plane of incidence, we note that $I_1 > I_2$, if $\sin 2\alpha < \sin 2\alpha'$, or, if B be the blackness, so that $B=1-I_1$, we have $B_1 > B_2$, if $\sin 2\alpha > \sin 2\alpha'$. We have seen that α and α' are zero at the critical angle, whence they increase to 90°, and that of the two α is the greater. Hence near the critical angle $2\alpha > 2\alpha'$, and both being less than 90° we have $\sin 2\alpha > \sin 2\alpha'$, so that $B_1 > B_2$, i.e. the dark spot is more conspicuous in the case of light polarised at right angles to the plane of incidence. This state of affairs lasts until α and α' are complementary, where

$$\sin 2\alpha = \sin 2\alpha', \text{ and } B_1 = B_2.$$

At this point the difference of phase between the components parallel and perpendicular to the plane of incidence is a minimum. The corresponding angle of incidence when $\mu=1{\cdot}596$ is $\phi=48^\circ\ 40'$. Beyond this incidence the order of magnitude of B_1 and B_2 is reversed, and the distinctness of the spots becomes more and more unequal as the incidence increases, and the spots contract. The following table gives the values of B_1 and B_2 for different angles of incidence, and different distances from the centre, when $\mu=1{\cdot}596$.

Stokes* obtained experimental results in complete harmony with this theory, although he did not find it possible to make a

* *Coll. Works*, Vol. II. p. 56.

sufficiently accurate measure of the intensity to render possible a numerical comparison of the results of theory and observation. The values of B_1 and B_2 given in the table below are shown

ϕ		$\rho=0$	$\rho=0{\cdot}5$	$\rho=1$	$\rho=1{\cdot}5$
39°	B_1	1	·962	·605	·211
	B_2	1	·808	·202	·042
40°	B_1	1	·949	·495	·092
	B_2	1	·800	·176	·022
45°	B_1	1	·846	·135	·002
	B_2	1	·766	·086	·001
50°	B_1	1	·687	·035	0
	B_2	1	·720	·041	0
60°	B_1	1	·342	·004	0
	B_2	1	·589	·010	0
75°	B_1	1	·057	0	0
	B_2	1	·245	·001	0
85°	B_1	1	·006	0	0
	B_2	1	·037	0	0

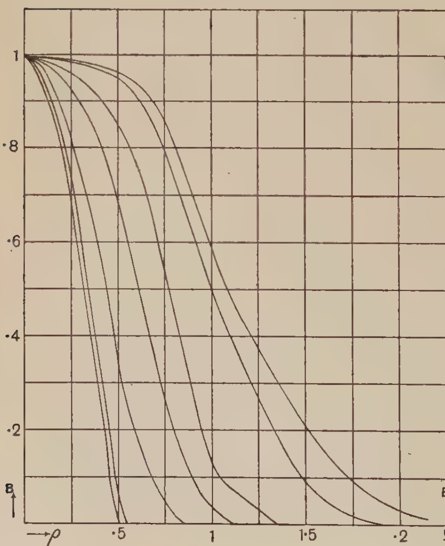

Fig. 42.

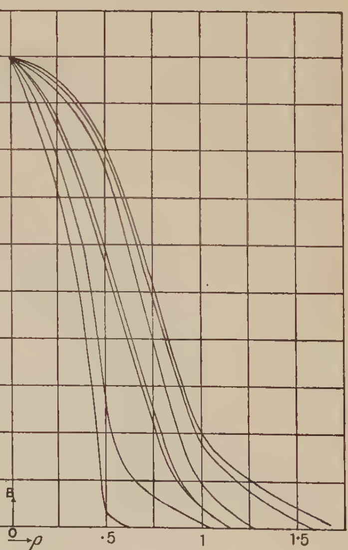

Fig. 43.

graphically in Figs. 42 and 43. These indicate clearly how the spot contracts in each case as the incidence rises, and how the relative intensity varies in the two cases.

It will have been observed that α, α', and q, on which the intensity depends, all involve μ, and so vary with the colour of the incident light. It would be easy to calculate the values of I for different values of μ, corresponding to the different constituent colours in the light used; but the factor that has a predominating influence on the intensity is q, and the variation of q depends chiefly on the presence of λ in the formula for q. Thus we may say, with sufficient accuracy for descriptive purposes, that $q = e^{-\kappa/\lambda}$, where κ is a constant. Hence q increases with λ, so that I diminishes. Thus the intensity of the blue constituent is

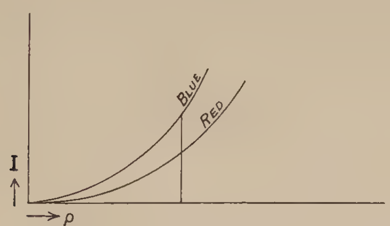

Fig. 44.

rather greater than that of the red, so that a faint bluish tint will be perceived round the dark spot—a deduction from theory that agrees with observation.

So far we have supposed that the ring system is new directly after the light is reflected or transmitted, as the case may be. Some interesting modifications of the results are obtained experimentally when the rings are viewed through an analyser, which destroys all vibrations except those confined to a particular plane. It is found convenient in practice to work with the transmitted system. We proceed to investigate the phenomena in this case from the point of view of our theory.

Let θ and θ' be the azimuths of the principal planes of the polariser and analyser respectively, represented by OE and OF in the figure. When the light emerges from the polariser the curl of the displacement is along OE' perpendicular to OE, and on passing

through the analyser it is along OF' perpendicular to OF. A wave of unit amplitude polarised parallel to the plane of incidence is transmitted through the plate as $R'e^{i\psi_1'}$, and if polarised at right angles to the plane of incidence it is transmitted as $Re^{i\psi_1}$, where

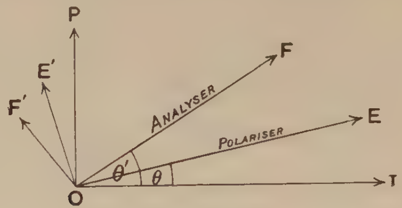

Fig. 45.

R, R', ψ_1, and ψ_1' are given by the theory already expounded. Hence a wave of unit amplitude polarised in an azimuth θ will emerge from the analyser as

$$R' \cos\theta \cos\theta' e^{i\psi_1'} + R \sin\theta \sin\theta' e^{i\psi_1},$$

and its intensity will be

$$I = R^2 \sin^2\theta \sin^2\theta' + R'^2 \cos^2\theta \cos^2\theta'$$
$$+ 2RR' \sin\theta \sin\theta' \cos\theta \cos\theta' \cos(\psi_1 - \psi_1').$$

The consideration of this formula, with the values of R, R', etc. given by the theory, will enable us to account for all the main features of the phenomena as revealed by experiment.

We have seen that the transmitted wave is represented by $\dfrac{qs_1s_2}{1+q^2r_1r_2}$, where $q = e^{-i\psi/2}$. It will make for brevity to write r for r_1r_2 and s for s_1s_2, and it will be noted that, on the theory of an abrupt transition, r and s are both real, s being positive, and r being positive, if the incidence is less than I_1 or greater than I_2, but otherwise negative. Here I_1 and I_2 denote, as on p. 97, the polarising angles at the two interfaces. For light polarised parallel to the plane of incidence, we replace r and s by r' and s', which are positive throughout.

For given values of θ and θ' it is obvious that I is a maximum or minimum along with R and R'. Thus the stationary points

occur, as before, when $\psi = n\pi$, and there is still a ring system. In general we have

$$\frac{qs}{1+q^2r} = \frac{se^{-i\psi/2}}{1+re^{-i\psi}} = \frac{se^{i\Phi}}{\sqrt{1+r^2+2r\cos\psi}} = Se^{i\Phi},$$

where $\Phi = \chi - \psi/2$, and

$$\tan\chi = (r\sin\psi)/(1+r\cos\psi).$$

At the centre $\psi = 0$, so that $\Phi = 0 = \Phi'$, and

$$S_0 = \frac{s}{1+r}, \quad S_0' = \frac{s'}{1+r'}.$$

At the next stationary point $\psi = \pi$, so that $\Phi = 0 = \Phi'$, and

$$S_1 = \frac{s}{1-r}, \quad S_1' = \frac{s'}{1-r'}.$$

Thus the difference of the intensities at these two stationary points, which determines the brightness of the rings, is

$$\begin{aligned} D &= [S_0\sin\theta\sin\theta' + S_0'\cos\theta\cos\theta']^2 \\ &\qquad - [S_1\sin\theta\sin\theta' + S_1'\cos\theta\cos\theta']^2 \\ &= -4\left[\frac{r^2s^2}{(1-r^2)^2}\sin^2\theta\sin^2\theta' + \frac{r'^2s'^2}{(1-r'^2)^2}\cos^2\theta\cos^2\theta'\right. \\ &\qquad \left. + \frac{ss'(r+r')\sin\theta\cos\theta\sin\theta'\cos\theta'}{(1-r^2)(1-r'^2)}\right]. \end{aligned}$$

We shall consider a few special cases of this formula. (1) When $\theta' = 90°$ we have

$$D = -4r^2s^2\sin^2\theta/(1-r^2)^2.$$

As this is negative, the rings are black centred. If we begin with $\theta = 90°$ the rings are most conspicuous at the outset, any turning of the polariser diminishing their brightness. The rings disappear completely when $\theta = 0$ or $180°$, for then the formula makes D vanish. (2) When $\theta' = 0$, we have

$$D = -4r'^2s'^2\cos^2\theta/(1-r'^2)^2.$$

The rings are still black centred throughout, except when they disappear, when the polariser is in the azimuth $\theta = 90°$ or $\theta = 270°$. (3) As an example of an intermediate azimuth of the analyser, take $\theta' = 45°$. We then have

$$\begin{aligned} D = &-2\,[r^2s^2\sin^2\theta/(1-r^2)^2 \\ &+ ss'(r+r')\sin\theta\cos\theta/(1-r^2)(1-r'^2) + r'^2s'^2\cos^2\theta/(1-r'^2)]. \end{aligned}$$

This is negative when $\theta = 0$ or $90°$. It vanishes, so that the rings disappear when

$$\tan\theta = -s'(1-r^2)/rs\,(1-r'^2),$$

and when $$\tan\theta = -r's'\,(1-r^2)/r^2 s\,(1-r'^2).$$

Between these two values of θ the rings are bright centred, whilst outside these limits they are dark centred.

At the polarising angles (I_1 or I_2) we have $r = 0$, and the formula for D reduces to

$$D = -\frac{4r's'\cos\theta\cos\theta'}{(1-r'^2)}\left[\frac{r's'}{(1-r'^2)}\cos\theta\cos\theta' + s\sin\theta\sin\theta'\right].$$

This vanishes when θ or θ' is $90°$, and also when

$$\tan\theta' = -\frac{r's'}{s\,(1-r'^2)}\cot\theta.$$

When $\theta' = 90°$, $D = 0$ and there are no rings. If θ' be slightly less than $90°$ the rings appear with their centres bright or dark, according as $\sin 2\theta$ is negative or positive. The rings disappear again at the azimuth θ' determined by

$$\tan\theta' = -r's'\cot\theta/s\,(1-r'^2),$$

and after this there is a change from one system of rings to the other. All these conclusions agree completely with observations on the behaviour of the rings under different circumstances.

When the angle of incidence is beyond the critical angle the intensity I is still given by the formula of p. 112, where R, R', ψ_1, ψ_1' have the values appropriate to the case of total internal reflection. The phenomena were examined experimentally by Stokes* with an angle of incidence slightly greater than the critical angle, and with the incident light plane polarised at an angle of $45°$ to the plane of incidence, and we shall consider this case more particularly. Putting $\theta = 45°$ in the formula for I, we get

$$I = \tfrac{1}{2}\left[R^2\sin^2\theta' + R'^2\cos^2\theta' + 2RR'\sin\theta'\cos\theta'\cos(\psi_1 - \psi_1')\right].$$

At the centre we have $R = R' = 1$, and $\psi_1 - \psi_1' = 0$, so that

$$I = \sin^2(\theta' + 45°).$$

Thus the intensity vanishes at the centre when $\theta' = 135°$. It also vanishes for all values of θ' at considerable distances from the centre where the thickness of the film is no longer very small, and

* *Math. and Physical Papers*, Vol. II. p. 56.

R and R' are inappreciable. I cannot vanish in the interval, but it may have certain maxima and minima as the tables and figures that follow show. The calculations have been made for $\mu = 1{\cdot}596$, and an angle of incidence of 39°, which is 13′ beyond the critical angle. The numbers represent the intensities for different

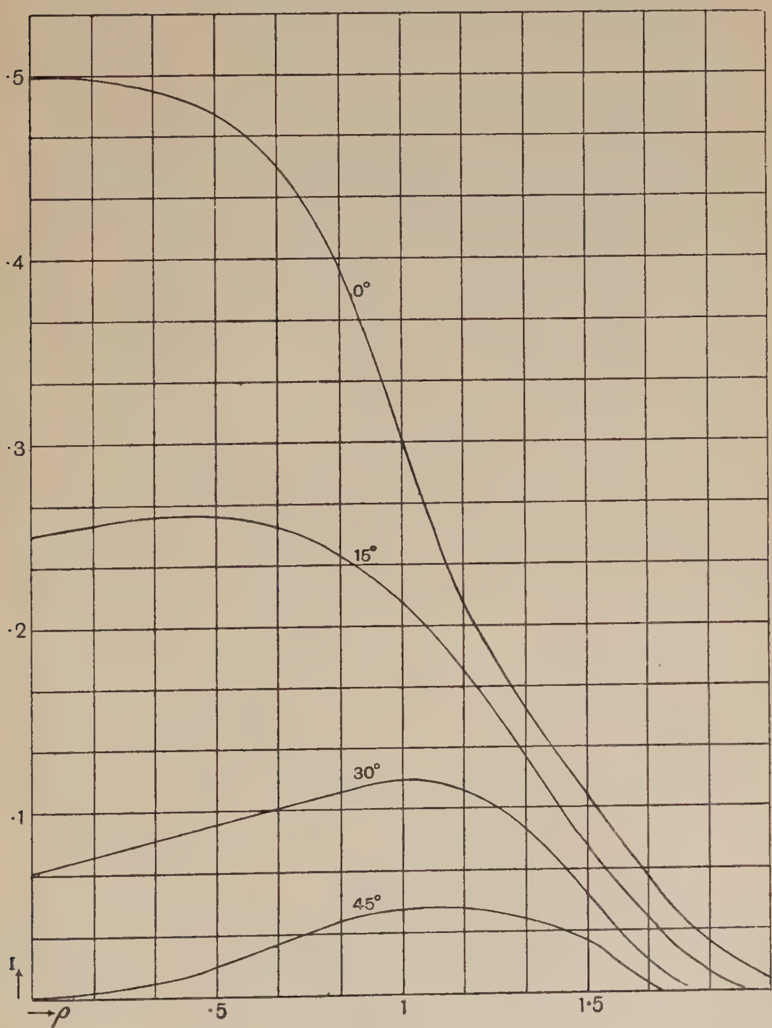

Fig. 46.

distances from the centre, and for different azimuths of the analyser (θ').

$\theta' - 90°$	$\rho=0$	$\rho=0·5$	$\rho=1$	$\rho=1·5$
0°	·5	·481	·303	·106
15°	·25	·263	·211	·083
30°	·067	·094	·117	·055
45°	0	·017	·045	·030

$\theta' - 90°$	$\rho=0$	$\rho=0·5$	$\rho=1$	$\rho=1·25$	$\rho=1·5$	$\rho=1·75$
45°	0	·017	·045	·042	·030	·009
50°	·008	·022	·030	·021	·023	·006
55°	·030	·030	·020	·013	·017	·004
60°	·067	·055	·016	·008	·013	·002

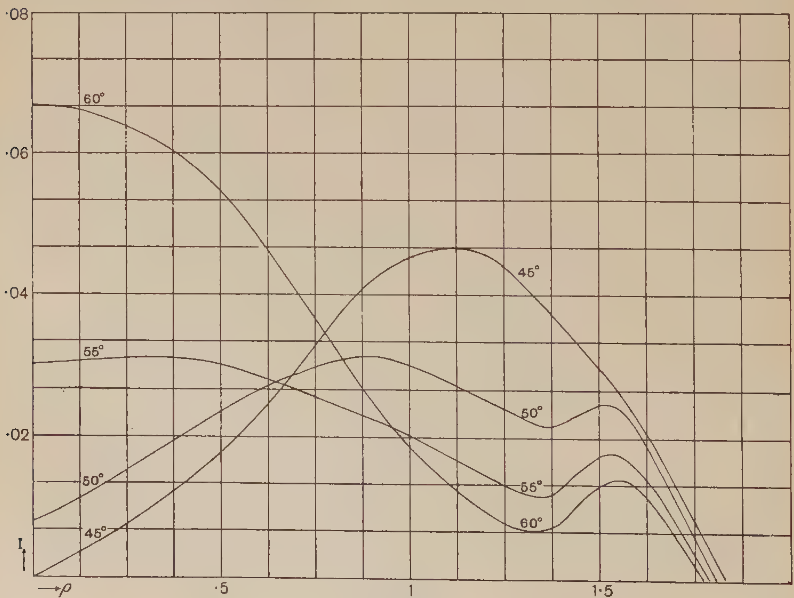

Fig. 47.

An examination of the figures will indicate clearly the nature of the phenomena. As θ' increases the bright spot at the centre becomes duller, and a bright ring is formed surrounding a central dull patch. The darkness of the centre increases, and the bright ring expands until at $\theta' = 135°$ the centre is absolutely black. After this the brightness of the centre increases, and a dark ring is formed which travels outwards until it is lost in the dark field outside. All this is in complete agreement with Stokes' observations.

The results so far obtained have been reached by means of Fresnel's formulae, which are strictly applicable only in the ideal case of an abrupt transition from one medium to another. The influence of the transition layer has been considered in the last chapter, whence it appears that the effect of the layer is to make r_1 and r_2 complex instead of real. We thus have $r_1 = R_1 e^{i\theta_1}$, and $r_2 = R_2 e^{i\theta_2}$, and as before $q^2 = Qe^{-i\psi}$, where R_1, R_2, and Q are real. Hence the intensity of the reflected beam, corresponding to

$$\frac{r_1 + q^2 r_2}{1 + q^2 r_1 r_2},$$

is
$$I = \frac{R_1^2 + Q^2 R_2^2 + 2QR_1R_2 \cos(\theta_2 - \psi - \theta_1)}{1 + Q^2 R_1^2 R_2^2 + 2QR_1R_2 \cos(\theta_2 - \psi + \theta_1)}.$$

We have seen that the departure from Fresnel's laws is very slight, except for light polarised at right angles to the plane of incidence in the immediate vicinity of Brewster's angle. This then will be the only region, where there can be any appreciable departure from the laws already reached as to the character and behaviour of the rings. As Brewster's angle is less than the critical angle we have $Q = 1$, and putting $R_1^2 + R_2^2 = a$, $1 + R_1^2 R_2^2 = a_1$, and $2R_1R_2 = b$, for brevity, the formula for I becomes

$$I = \frac{a + b\cos(\theta_2 - \psi - \theta_1)}{a_1 + b\cos(\theta_2 - \psi + \theta_1)}.$$

For a given angle of incidence ψ is the only variable in this expression, so that I is a periodic function whose period is 2π. There will thus be alternatives of bright and dark rings, the stationary positions of I being given by the equation:

$$(a - a_1)\cos\theta_1 \sin(\theta_2 - \psi) + (a + a_1)\sin\theta_1 \cos(\theta_2 - \psi) + b\sin 2\theta_1 = 0.$$

Putting
$$\tan\beta = \frac{a_1 + a}{a_1 - a}\tan\theta_1,$$
and
$$\sin\gamma = \frac{b\sin 2\theta_1}{\sqrt{(a_1 - a)^2 + 4aa_1\sin^2\theta_1}},$$
we get $\sin(\psi + \beta - \theta_2) = \sin\gamma$, so that $\psi + \beta - \theta_2 = n\pi + (-1)^n\gamma$.

For light polarised in the plane of incidence θ_1 and θ_2 are very nearly zero or π throughout, so that β and γ are very small or nearly π, and $\psi = m\pi$ very approximately, where m is any integer. This is the formula obtained on the hypothesis of an abrupt transition, so that the layer has a very slight influence on either the size or the intensity of the rings.

For light polarised at right angles to the plane of incidence $b\sin 2\theta_1$ is always small, for $\sin 2\theta_1$ is very small except near the Principal Incidence, and in this neighbourhood b is very small. Hence γ is always very small, and we have very approximately
$$\psi + \beta - \theta_2 = n\pi.$$
The intensity I is a maximum when n is even, in which case its value is
$$\frac{a + b\cos(\theta_1 - \beta)}{a_1 + b\cos(\theta_1 + \beta)},$$
while its minimum value occurs when n is odd, and is
$$\frac{a - b\cos(\theta_1 - \beta)}{a_1 - b\cos(\theta_1 + \beta)}.$$
Owing to the smallness of b near the Principal Incidence, the variations of the intensity are very slight, so that the rings in this neighbourhood are faint, although not absolutely evanescent, as would be the case with an abrupt transition.

From the foregoing discussion it will be apparent that the layer influences the phenomena mainly in five ways:

1. The centre is no longer one of the positions of maxima or minima intensity. At the centre we have $\psi = 0$, and
$$I = \frac{a + b\cos(\theta_2 - \theta_1)}{a_1 + b\cos(\theta_2 + \theta_1)}.$$

Near the first Principal Incidence (I_1), θ_2 is nearly zero or π, so that the intensity at the centre is

$$I = \frac{a \pm b \cos \theta_1}{a_1 \pm b \cos \theta_1}$$

approximately. At the stationary points, on the other hand, we have

$$I = \frac{a \pm b \cos (\theta_1 - \beta)}{a_1 \pm b \cos (\theta_1 + \beta)},$$

and owing to the presence of β this is not identical with the intensity at the centre. There will be very little difference between the two intensities, except in the neighbourhood of the Principal Incidence, for outside that region β is very nearly zero or π.

2. As b is not zero at the polarising angle the rings do not disappear there, although they are very faint in this region, owing to the smallness of b.

3. The size of the rings is somewhat altered. We have seen that the radii are given by the formula

$$\rho = r \sqrt{\kappa_1/\lambda} = \sqrt{(\sec \phi') \psi/\pi}.$$

As ψ is no longer exactly equal to $n\pi$, there is a change in the radii. This change, however, is appreciable only in the neighbourhood of the Principal Incidence, when we have $\psi + \beta = n\pi$ very nearly and

$$\rho = \sqrt{(n - \beta/\pi) \sec \phi'}.$$

4. The rings do not all expand steadily when the angle of incidence increases, as is the case with an abrupt transition; but, under certain conditions, they experience alternations of expansion and contraction. The formula for ρ shows that

$$2\pi\rho \frac{d\rho}{d\phi'} = \sec \phi' \left[(n\pi - \beta) \tan \phi' - \frac{d\beta}{d\phi'} \right].$$

If β diminishes with the incidence, $\frac{d\beta}{d\phi'}$ is negative and ρ increases continuously with ϕ'. If, however, $\frac{d\beta}{d\phi'}$ be positive, the rings will be stationary at the incidences determined by

$$\frac{d\beta}{d\phi'} = (n\pi - \beta) \tan \phi'.$$

The equation for β shows that the graph of β will be very similar to that of θ_1, the gradient being smaller. Thus $\frac{d\beta}{d\phi'}$ is very small until the Principal Incidence is approached, when it rises rapidly and falls again equally rapidly almost to zero. The graph of such a function (see Fig. 48) will show that the equation

$$\frac{d\beta}{d\phi'} = (n\pi - \beta) \tan \phi',$$

has one real root, two real, or none, the region of unreal roots being approached as n increases, i.e. as the order of the rings increases. In the case of real roots ϕ_1' and ϕ_2', the rings will expand until $\phi' = \phi_1'$, then contract until $\phi' = \phi_2'$, and beyond that expand. If there are no real roots the rings continually expand as the incidence rises.

We have seen that the contraction of the rings near the Principal Incidence cannot take place for the larger rings unless $\frac{d\beta}{d\phi'}$ is very large, and that it cannot occur at all unless $\frac{d\beta}{d\phi'}$ is positive. To give more definiteness to the discussion, let us suppose that the second medium is less refractive than the other two as will be the case, for example, with air between two pieces of glass, or air between glass and diamond. From the formula for an abrupt transition

$$r_1 = -\tan(\phi - \phi')/\tan(\phi + \phi'),$$

it appears that, since $\phi' > \phi$, r_1 is positive until the Principal Incidence I_1 is reached, after which it is negative. If then we put $r_1 = R_1 e^{i\theta_1}$ we shall have θ_1 zero up to I_1, and then a sudden change to $\theta_1 = \pi$. The effect of the layer is to make this change in θ_1 gradual, so that θ_1 (and therefore also β) increases continuously from 0 to π. Thus $\frac{d\beta}{d\phi'}$ is positive, and the contraction of the rings of lower order is to be expected in the neighbourhood of the Principal Incidence.

We have dealt specially with the first Principal Incidence (I_1), but very little modification is needed when dealing similarly with the neighbourhood of I_2. In this region we have $\psi - \theta_2 = n\pi$ very nearly, so that we have merely to replace β by $-\theta_2$. In the

special case mentioned above θ_2 goes from π to zero in passing through I_2, so that $-\frac{d\theta_2}{d\phi'}$ is positive, and the same phenomena are to be expected near I_2 as near I_1.

By way of a numerical example we shall take the case where the third medium is diamond and the second air, and consider the nature of the phenomena in the neighbourhood of I_2. The equation determining the stationary points of the rings is

$$n+\frac{\theta_2}{\pi}=-\frac{\cot\phi'}{\pi}\frac{d\theta_2}{d\phi'}.$$

The angle θ_2 is the ρ of p. 74, and is given by the formula

$$\tan\theta_2=\frac{4\epsilon_1\mu^4\sec\phi'}{(1+\mu^2)^{\frac{3}{2}}(\tan^2\phi'-\mu^2)}.$$

Thus we get

$$-\cot\phi'\frac{d\theta_2}{d\phi'}=\frac{(1+\mu^2)^{\frac{3}{2}}}{4\epsilon_1\mu^4}\sin^2\theta_2\,[\sec\phi'+(1+\mu^2)\cos\phi'],$$

from which it appears that $-\cot\phi'\frac{d\theta_2}{d\phi}$ is a maximum at the Principal Incidence, where $\tan\phi'=\mu$, and its value is then $\frac{1}{2\epsilon_1}(1+1/\mu^2)^2$. Hence the greatest value of n for rings that experience any contraction is given by

$$n+1/2=(1+1/\mu^2)^2/2\pi\epsilon_1.$$

Taking $\mu=2{\cdot}434$, and $\epsilon_1=0{\cdot}0241$ for diamond we find that n cannot be greater than eight, so that there are only nine contracting rings, five of them being dark. The following table gives the values of $\frac{\theta_2}{\pi}$ and $\frac{-\cot\phi'}{\pi}\frac{d\theta_2}{d\phi'}$, for various incidences.

ϕ'	60°	65°	66°	67°	$I_2=67°\ 40'$	68°	69°	70°	75°
θ_2/π	·935	·846	·780	·647	·5	·420	·243	·156	·045
$-\frac{\cot\phi'}{\pi}\frac{d\theta_2}{d\phi'}$	·23	1·24	2·30	4·55	9·01	5·33	2·72	1·26	·12

A graph of these functions enables us to determine the

stationary points of the rings, and the regions within which they contract.

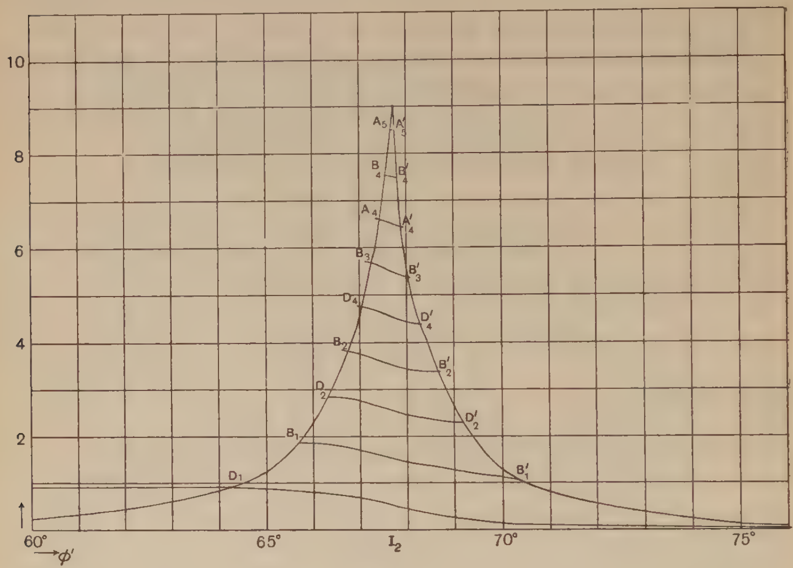

Fig. 48.

In Fig. 48 the curve $BDB'D'$ represents the function $\dfrac{-\cot\phi'}{\pi}\dfrac{d\theta_2}{d\phi'}$, and the curves B_nB_n', D_nD_n' represent $n+\theta_2/\pi$, for odd and even values of n respectively, so that B_nB_n' corresponds to a bright ring, and D_nD_n' to a dark one. The points of intersection of the curves indicate the incidences, where the rings are stationary and the contracting process begins or ends. The first bright ring corresponds to $n=0$, and it appears from the figure that once the contraction begins it continues until the incidence is grazing. The region of contraction diminishes as the order of the ring increases, and beyond the ninth ring there is no contraction at all.

(5) The rings do not change suddenly from a dark to a light centred system on passing through the Principal Incidence; but the transition from one system to the other is effected gradually. Thus with the air-diamond combination discussed under (4) the rings are dark centred to begin with at normal incidence, and expand

as ϕ' increases. As we approach I_1 the bright ring surrounding the centre contracts until it squeezes out the dark centre, so that after passing I_1 the rings are bright centred. Then as I_2 is approached, the dark ring surrounding the centre contracts, and continues this process until the incidence is grazing. Thus the bright centre is soon squeezed out, and the rings appear to be dark centred.

When considering the arrangement of colours in Newton's rings, we observed that for the rings of higher order, where the thickness of the films is a few wave lengths, the overlapping of the different coloured rings is so great that the interference effects are not observable. Thus for bright colours to appear, the thickness of the film must be of the order of magnitude of a wave length. However it should be noted that the fundamental formula from which all the results of this chapter have been derived is true whether the transparent plate or film be thin or thick. The formula referred to is that obtained on p. 95, where it is shown that the reflected beam is represented by

$$(r_1 + q^2 r_2)/(1 + q^2 r_1 r_2),$$

which reduces to

$$r(1 - q^2)/(1 - q^2 r^2)$$

when the first and third media are the same. We shall have occasion to apply this formula in cases where the plate is not thin. We have seen that below the critical angle q is given by the formula

$$q^2 = e^{-i\psi} = e^{-2ix},$$

where
$$x = \psi/2 = \frac{\pi}{\lambda} \cdot \frac{\mu'}{\mu} c_1 \cos\phi'.$$

Hence we have

$$\left|\frac{r(1-q^2)}{1-q^2r^2}\right|^2 = \frac{r^2(1-\cos 2x)}{1+r^4-2r^2\cos 2x} = \frac{4r^2\sin^2 x}{(1-r^2)^2+4r^2\sin 2x}.$$

This formula refers to homogeneous light, of a definite wave length, otherwise λ, μ, ϕ' and therefore r and x would not be definite quantities. If natural light be employed, we have to deal with different wave lengths, and r and x will be different for the various constituents. The variation in r will usually, however, be small; but x will vary enormously for a plate of ordinary thick-

ness, for which c_1/λ is very great. Hence the intensity of the reflected light may be obtained in this case (when the incident light is natural light of unit intensity) by regarding r as constant, and giving x all possible values from 0 to π. Thus we find that the intensity is

$$R^2 = \frac{1}{\pi}\int_0^\pi \frac{4r^2 \sin^2 x \, dx}{(1-r^2)^2 + 4r^2 \sin^2 x} = \frac{2r^2}{1+r^2}.$$

The sum of the intensities of the different constituents of the reflected beam (see Fig. 34) is easily seen to be

$$r^2 + r^2(1-r^2)^2[1 + r^4 + r^6 + \ldots] = 2r^2/(1+r^2).$$

Hence the intensity of the reflected beam is the sum of the intensities of its constituents.

The result just obtained enables us to estimate the intensity of the reflected and transmitted beams when natural light is incident on a pile of transparent plates, and to measure the degree of polarisation of the transmitted light. We shall suppose the plates to be all of the same material and thickness, and too thick to give rise to the colours of thin films. Let R_n and T_n denote the intensities of the reflected and transmitted light when there are n plates. If we add one more plate behind the pile, the quantity T_{n+1} is made up of the light T_n after this has been transmitted through the single plate once, twice,... after successive reflections from the pile, and in calculating T_{n+1} we have merely to find the sum of the intensities of the various constituents. Thus we get

$$T_{n+1} = T_1 T_n (1 + R_1 R_n + R_1^2 R_n^2 + \ldots) = T_1 T_n/(1 - R_1 R_n).$$

Moreover since the plates are perfectly transparent there is no loss of light, so that $T_n + R_n = 1$. Hence

$$T_{n+1} = T_1 T_n/(T_1 + R_1 T_n), \text{ or } 1/T_{n+1} = 1/T_n + R_1/T_1.$$

Solving this difference equation, and determining the constant by means of the relation $R_1 + T_1 = 1$, we get $1/T_n = 1 + nR_1/T_1$. Hence

$$T_n = T_1/(T_1 + nR_1), \text{ and } R_n = 1 - T_n = nR_1/(T_1 + nR_1).$$

It thus appears that when n is infinite $T_n = 0$ and $R_n = 1$, so that all the light is reflected. This helps to explain the whiteness of

snow, and other finely divided substances that are transparent in mass.

The formulae for T_n and R_n just obtained give the intensity in terms of the number of plates, and the intensities R_1 and T_1 of the reflected and transmitted light when there is only one plate. For a single plate we have

$$R_1 = R^2 = 2r^2/(1+r^2), \text{ and } T_1 = 1 - R_1 = (1-r^2)/(1+r^2)$$

where r is the r or r' of Fresnel's formulae, according as the incident light is polarised at right angles or parallel to the plane of incidence. The following table, due to Stokes*, gives the intensities at normal incidence, at the polarising angle $I = \tan^{-1}\mu$, and at $\phi = I + 2°$ when the refractive index is $\mu = 1{\cdot}52$.

n	$\phi=0$		$\phi=I$		$\phi=I+2°$			
	$R_n=R_n'$	$T_n=T_n'$	R_n'	T_n'	R_n'	T_n'	R_n	T_n
1	·082	·918	·271	·729	·300	·700	·001	·999
2	·151	·849	·426	·574	·459	·541	·002	·998
4	·262	·738	·598	·402	·628	·372	·004	·996
8	·416	·584	·749	·251	·771	·229	·008	·992
16	·587	·413	·856	·144	·870	·130	·016	·984
32	·740	·260	·922	·078	·931	·069	·032	·968
∞	1	0	1	0	1	0	1	0

We have seen in Chap. II. that natural light of unit intensity can be represented by two streams, each of intensity 1/2, polarised respectively at right angles and parallel to the plane of incidence. The intensities of the transmitted streams are $\frac{1}{2}T_n$ and $\frac{1}{2}T_n'$, where

$$T_n = \frac{T_1}{T_1 + nR_1} = \frac{1 - R_1}{1 + (n-1)R_1} = \frac{1 - r^2}{1 + (2n-1)r^2},$$

and
$$T_n' = \frac{1 - r'^2}{1 + (2n-1)r'^2}.$$

For brevity we shall put

$$\phi - \phi' = \rho, \quad \phi + \phi' = \sigma, \quad \operatorname{cosec}^2\rho = c_1, \quad \text{and} \quad \operatorname{cosec}^2\sigma = c_2,$$

* *Math. and Physical Papers*, Vol. IV. p. 152.

then from Fresnel's formulae we have

$$r^2 = \tan^2 \rho/\tan^2 \sigma = (c_2 - 1)/(c_1 - 1),$$

and $$r'^2 = \sin^2 \rho/\sin^2 \sigma = c_2/c_1.$$

Hence denoting the degree of polarisation of the transmitted light by P, we have

$$P = \frac{T_n - T_n'}{T_n + T_n'} = \frac{n(r'^2 - r^2)}{1 + (n-1)(r'^2 + r^2) - (2n-1) r^2 r'^2}$$

$$= \frac{n[c_2/c_1 - (c_2-1)/(c_1-1)]}{1 + (n-1)[c_2/c_1 + (c_2-1)/(c_1-1)] - (2n-1) c_2 (c_2-1)/c_1 (c_1-1)}$$

$$= \frac{n}{c_1 + (2n-1) c_2 - n} = \frac{n}{\operatorname{cosec}^2 \rho + (2n-1)\operatorname{cosec}^2 \sigma - n}.$$

The following table gives the values of P for different values of n and ϕ, when $\mu = 1{\cdot}52$, the results being represented graphically in Fig. 49. If we wish to determine the maximum value of P we

n	$\phi = 10°$	$30°$	$50°$	$60°$	$75°$
1	·003	·034	·114	·182	·320
2	·006	·063	·203	·307	·440
4	·011	·109	·334	·470	·544
8	·018	·172	·493	·637	·614
16	·025	·238	·647	·775	·658
32	·031	·300	·770	·870	·681
∞	·042	·402	·943	·988	·707

must make $\operatorname{cosec}^2 \rho + (2n-1)\operatorname{cosec}^3 \sigma$ a minimum. For this purpose we note that, since $\sin \phi = \mu \sin \phi'$, we have

$$\frac{d\phi}{\tan \phi} = \frac{d\phi'}{\tan \phi'} = \frac{d\rho}{\tan \phi - \tan \phi'} = \frac{d\sigma}{\tan \phi + \tan \phi'}$$

$$= \cos \phi \cos \phi' \, . \, d\theta \text{ (say)}.$$

Hence $$d\rho = \sin \rho \, d\theta, \quad \text{and} \quad \sin \sigma \, . \, d\theta.$$

Thus P is a maximum when

$$\operatorname{cosec}^2 \rho \cos \rho + (2n-1) \operatorname{cosec}^2 \sigma \cos \sigma = 0,$$

and its maximum value is

$$P_1 = \frac{2\mu^2}{(1+\mu^2)\sin\phi} - 1.$$

The equation giving the position of the maximum may be written in the form

$$2n - 1 = -\frac{\operatorname{cosec}^2\rho \cdot \cos\rho}{\operatorname{cosec}^2\sigma \cdot \cos\sigma} = -\frac{1}{rr'}.$$

Since ρ is less than $90°$, $\cos\rho$ is positive, so that as $2n-1$ is positive we must have $\cos\sigma$ negative. Thus $\phi + \phi'$ is greater

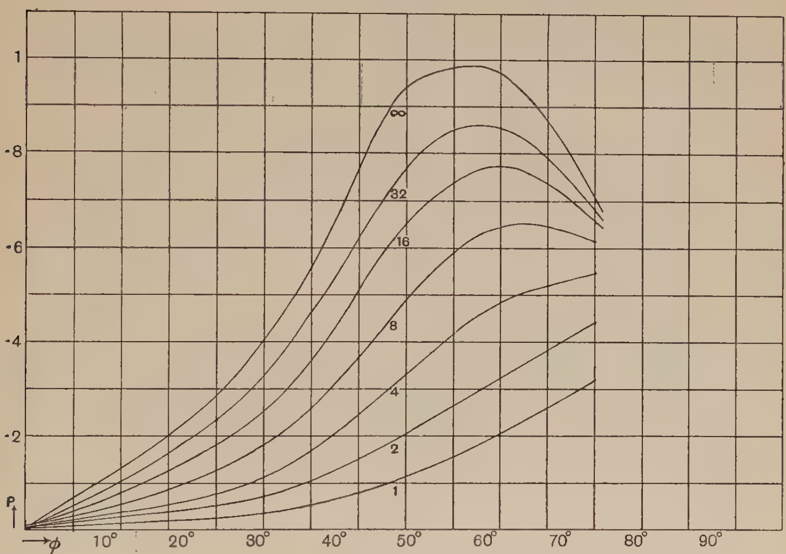

Fig. 49.

than $90°$, and the angle of incidence is beyond Brewster's angle. We have seen that the numerical values of r and r' increase with the angle of incidence from Brewster's angle to grazing incidence. Hence in this range $-1/rr'$ diminishes, and consequently n decreases from infinity to unity. At the same time the maximum polarisation diminishes from 1 to $(\mu^2-1)/(\mu^2+1)$. We see then that as the number of plates increases the polarisation tends to become complete, and the angle of incidence that gives the greatest degree of polarisation approaches more and more nearly to Brewster's angle.

The following table gives corresponding values of n, P_1, and ϕ calculated from the formulae

$$P_1 = \frac{2\mu^2}{(1+\mu^2)\sin^2\phi} - 1,$$

and

$$2n - 1 = -\frac{\operatorname{cosec}^2\rho\cos\rho}{\operatorname{cosec}^2\sigma\cos\sigma}$$

for the case already considered where $\mu = 1{\cdot}52$.

ϕ	56° 40′	60°	65°	70°	75°	80°	85°	90°
n	∞	30·372	9·775	4·921	2·913	1·944	1·330	1
P_1	1	·925	·823	·735	·663	·610	·578	·567

By representing these results graphically, as in Fig. 50, we can calculate approximately the angles of incidence that give the greatest degree of polarisation for any integral value of n, and also obtain a numerical measure of the polarisation at that incidence.

This figure gives the graphs both of P_1 and n, the former corresponding to the dotted, and the latter to the continuous line. The scale of n is indicated on the left, and that of P_1 on the right. The values of P_1 and ϕ estimated from these graphs for different integral values of n are set out in the following table.

n	1	2	3	4	5	6	7
ϕ	90°	79° 30′	74° 45′	72°	69° 45′	68° 15′	67° 15′
P_1	·567	·615	·667	·707	·739	·763	·783
n	8	9	10	15	30	∞	
ϕ	66° 20′	65° 30′	64° 45′	63°	60° 5′	56° 40′	
P_1	·798	·814	·827	·862	·922	1	

The consideration of some other properties of thick plates suggested by the subject matter of this chapter will be more

conveniently postponed until the theory of dispersion has been discussed.

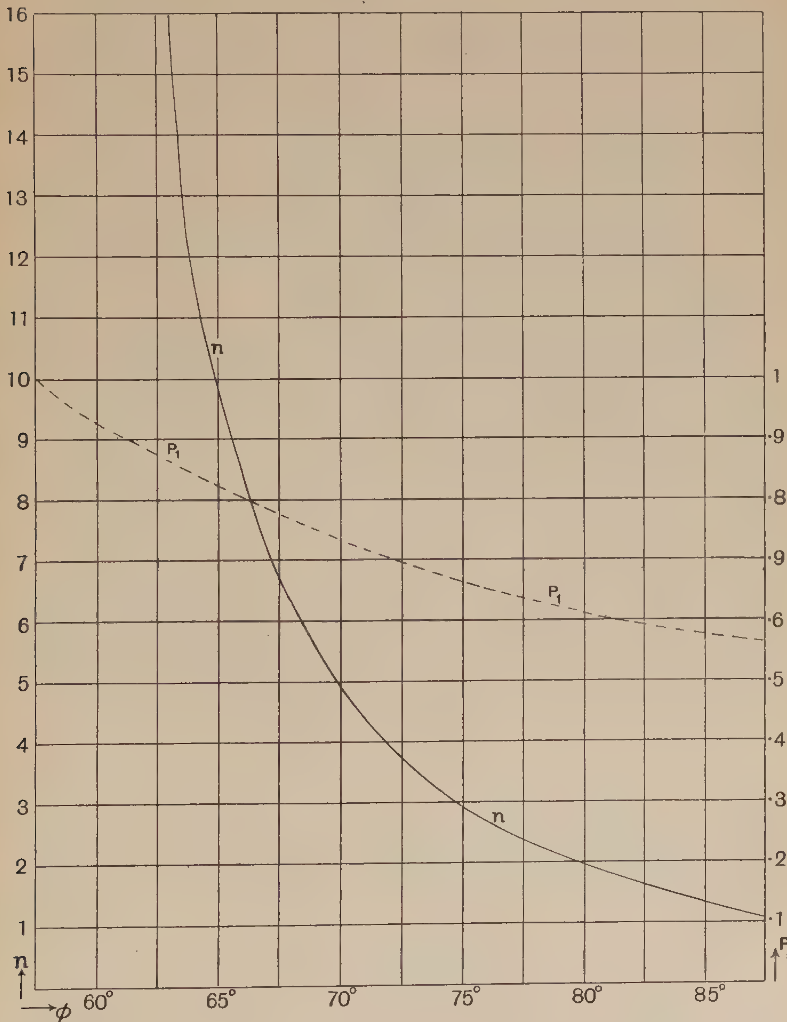

Fig. 50.

CHAPTER VI.

PROPAGATION OF LIGHT IN TRANSPARENT CRYSTALLINE MEDIA.

WHEN dealing in Chapter III. with the propagation of light in a material medium, we found it convenient mentally to replace the matter by a continuum formed by smoothing out the atoms, and spreading their influence uniformly throughout the body. It was tacitly assumed that the material molecules were arranged perfectly at random, so that the smoothed out representative medium was isotropic. Experience, however, shows us in the case of crystals that for matter of some kinds the molecules are not arranged at random. Under such circumstances the representative medium will not be the same in all directions. Hence it will no longer be possible to express the potential energy per unit volume in terms of a single constant c/μ, and write W_1 in the form

$$\frac{1}{2}\frac{c^2}{\mu^2}(f^2+g^2+h^2).$$

W_1 will now be a general quadratic function of f, g, h. This function will have certain planes of symmetry, and by taking the axes of symmetry as coordinate axes we can express W_1 in the form

$$\frac{1}{2}(a^2f^2+b^2g^2+c^2h^2).$$

The investigation of the propagation of light in such a medium can be carried out on exactly the same lines as before with the aid of the Principle of Action, and the formulae

$$T=\frac{1}{2}\int(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)\,d\tau,$$

and

$$W=\frac{1}{2}\int(a^2f^2+b^2g^2+c^2h^2)\,d\tau.$$

The dynamical equations are found as on p. 33, and prove to be

$$\ddot{\xi} = b^2\frac{\partial g}{\partial z} - c^2\frac{\partial h}{\partial y}, \quad \ddot{\eta} = c^2\frac{\partial h}{\partial x} - a^2\frac{\partial f}{\partial z}, \quad \ddot{\zeta} = a^2\frac{\partial f}{\partial y} - b^2\frac{\partial g}{\partial x};$$

while the surface conditions require the continuity of

$$nb^2g - mc^2h, \quad lc^2h - na^2f, \quad \text{and} \quad ma^2f - lb^2g,$$

and of course that of ξ, η, ζ. These boundary conditions may also be expressed by saying that the displacement (ξ, η, ζ) is continuous at an interface, as are also the tangential components of the vector (a^2f, b^2g, c^2h).

As we have to deal throughout with periodic functions of the time we may put everything proportional to e^{ipt}, and so replace $\ddot{\xi}$ by $-p^2\xi$, and so for $\ddot{\eta}$ and $\ddot{\zeta}$. The dynamical equations then show that ξ, η, ζ may be regarded as the curl of the vector

$$\frac{1}{p^2}(a^2f, \ b^2g, \ c^2h).$$

Hence
$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0,$$

so that there is no compression involved in the motion.

If (ξ_1, η_1, ζ_1) be the vector of which (ξ, η, ζ) is the curl we have $\xi_1 = fa^2/p^2$, and so for η_1 and ζ_1. The dynamical equations may be written in terms of this new vector in the form

$$\ddot{\xi}_1 = -a^2f, \quad \ddot{\eta}_1 = -b^2g, \quad \ddot{\zeta}_1 = -c^2h.$$

But we have

$$f = \frac{\partial \zeta}{\partial y} - \frac{\partial \eta}{\partial z} = -\nabla^2\xi_1 + \frac{\partial}{\partial x}\left(\frac{\partial \xi_1}{\partial x} + \frac{\partial \eta_1}{\partial y} + \frac{\partial \zeta_1}{\partial z}\right).$$

Hence
$$\ddot{f} = a^2\nabla^2 f - \frac{\partial \theta}{\partial x}, \quad \ddot{g} = b^2\nabla^2 g - \frac{\partial \theta}{\partial y}, \quad \ddot{h} = c^2\nabla^2 h - \frac{\partial \theta}{\partial z},$$

where
$$\theta = a^2\frac{\partial f}{\partial x} + b^2\frac{\partial g}{\partial y} + c^2\frac{\partial h}{\partial z}.$$

We shall consider a progressive plane wave moving in the direction (l, m, n) with velocity v. Then we may take

$$(\xi, \eta, \zeta) = (L, M, N) \,.\, Ae^{i\kappa'(lx+my+nz-vt)} = (L, M, N)\, Ae^{i\omega} \text{ say},$$

and
$$(f, g, h) = (\lambda, \mu, \nu)\, Be^{i(\omega - \pi/2)},$$

where L, M, N are the direction cosines of the displacement, and

λ, μ, ν those of the curl, while A and B are the amplitudes of these two respectively. Since

$$\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}+\frac{\partial \zeta}{\partial z}=0,$$

we have $$lL+mM+nN=0 \quad\text{.......................(I)},$$

so that the displacement is in the wave front. Also since

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}=0,$$

we have $$l\lambda+m\mu+n\nu=0 \quad\text{.......................(II)},$$

so that the curl is also in the wave front. Substituting for f, g, h in the equations

$$\ddot{f}=a^2\nabla^2 f-\frac{\partial \theta}{\partial x} \text{ etc.},$$

we get $$\lambda(v^2-a^2)+l(a^2 l\lambda+b^2 m\mu+c^2 n\nu)=0,$$

and two similar equations. Hence we have

$$\frac{l}{v^2-a^2}=-\frac{\lambda}{\kappa},\quad \frac{m}{v^2-b^2}=-\frac{\mu}{\kappa},\quad \frac{n}{v^2-c^2}=-\frac{\nu}{\kappa} \quad\text{......(III)},$$

where $$\kappa=a^2 l\lambda+b^2 m\mu+c^2 n\nu.$$

Multiplying the first of these by l, the second by m, and the third by n, we get on addition by means of (II)

$$\frac{l^2}{v^2-a^2}+\frac{m^2}{v^2-b^2}+\frac{n^2}{v^2-c^2}=0 \quad\text{...........(IV)}.$$

Again multiplying

$$\lambda(v^2-a^2)+l(a^2 l\lambda+b^2 m\mu+c^2 n\nu)=0$$

by λ, and the other two similar equations by μ and ν, and adding, we get

$$v^2=a^2\lambda^2+b^2\mu^2+c^2\nu^2 \quad\text{.....................(V)}.$$

Finally from (III) we have

$$\frac{l}{\lambda}=-\frac{1}{\kappa}(v^2-a^2),\quad \frac{m}{\mu}=-\frac{1}{\kappa}(v^2-b^2),\quad \frac{n}{\nu}=-\frac{1}{\kappa}(v^2-c^2),$$

hence $$\frac{l}{\lambda}(b^2-c^2)+\frac{m}{\mu}(c^2-a^2)+\frac{n}{\nu}(a^2-b^2)=0 \quad\text{.........(VI)}.$$

These six equations (I—VI) enable us to deal completely with the problem of the propagation of light in transparent crystals.

It will be convenient at the outset to find the relation between

A and B the amplitudes of the displacement and its curl, and to obtain a formula for the kinetic and potential energies in terms of these amplitudes. Putting $\kappa' v = p$, so that $p/2\pi$ is the frequency, and substituting for (ξ, η, ζ) and (f, g, h) in the dynamical equation

$$\ddot{\xi} = b^2 \frac{\partial g}{\partial z} - c^2 \frac{\partial h}{\partial y},$$

we get $$p^2 AL = B\kappa' (c^2 m\nu - b^2 n\mu).$$

But since the lines (l, m, n), (λ, μ, ν), and (L, M, N) are rectangular, we have $l = \mu N - \nu M$, $m = \nu L - \lambda N$, and $n = \lambda M - \mu L$.

Hence $$c^2 m\nu - b^2 n\mu = c^2 \nu (\nu L - \lambda N) - b^2 \mu (\lambda M - \mu L)$$
$$= L (a^2\lambda^2 + b^2\mu^2 + c^2\nu^2) - \lambda (a^2\lambda L + b^2\mu M + c^2\nu N).$$

By (V) we have

$$a^2\lambda^2 + b^2\mu^2 + c^2\nu^2 = v^2,$$

also

$$a^2\lambda L + b^2\mu M + c^2\nu N = a^2\lambda (m\nu - n\mu) + b^2\mu (n\lambda - l\nu) + c^2\nu (l\mu - m\lambda)$$
$$= \lambda\mu\nu \left[\frac{l}{\lambda}(b^2 - c^2) + \frac{m}{\mu}(c^2 - a^2) + \frac{n}{\nu}(a^2 - b^2)\right]$$
$$= 0 \text{ by (VI)}.$$

Hence $$c^2 m\nu - b^2 n\mu = Lv^2,$$

and $$p^2 AL = B\kappa' L v^2 = pvBL.$$

Thus $A = (v/p)B$, which is the relation between amplitudes of the displacement and its curl.

The kinetic energy per unit volume is

$$\tfrac{1}{2} (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) = -\tfrac{1}{2} p^2 A^2 e^{2i\omega}.$$

The potential energy is similarly

$$\tfrac{1}{2} (a^2 f^2 + b^2 g^2 + c^2 h^2) = \tfrac{1}{2} (a^2\lambda^2 + b^2\mu^2 + c^2\nu^2) B^2 e^{i(2\omega - \pi)}$$
$$= \tfrac{1}{2} v^2 B^2 e^{i(2\omega - \pi)} = -\tfrac{1}{2} p^2 A^2 e^{2i\omega}.$$

Thus the kinetic energy is equal to the potential, and the intensity is proportional to A^2, whichever method of measuring the intensity be adopted.

Equation (IV) above, viz.

$$l^2/(v^2 - a^2) + m^2/(v^2 - b^2) + n^2/(v^2 - c^2) = 0,$$

gives the velocity of propagation of a wave in any given direction. As it is a quadratic equation in v^2 there are in general two distinct waves corresponding to any wave normal, the two waves travelling at different speeds. Before dealing with the most general case we shall consider the simpler one where $b = c$. We then have

$$l^2/(v^2 - a^2) + (1 - l^2)/(v^2 - c^2) = 0, \text{ since } l^2 + m^2 + n^2 = 1.$$

This leads at once to

$$v^2 = c^2, \text{ or } v^2 = l^2c^2 + a^2(1 - l^2).$$

The first root makes v a constant, so that for this case the phenomena are the same as in an isotropic medium, and the ordinary laws of refraction apply. The wave is therefore called the ordinary wave. The alternative root is $v^2 = l^2c^2 + a^2(1 - l^2)$, so that in this case v depends on the direction of propagation, and the ordinary laws of refraction do not hold any longer. The corresponding wave is consequently styled the extraordinary wave.

If, in the expressions just obtained for v^2, we make the substitutions $v = 1/r$, $l = x/r$, $m = y/r$, and $n = z/r$, we get $r = 1/c$ for the ordinary wave, and $1 = a^2(y^2 + z^2) + c^2x^2$ for the extraordinary one. These equations represent the two sheets of a surface called the *Index Surface* from the fact that the radius vector in any direction is proportional to the refractive index for a wave, whose normal is in that direction. It is at once evident that the Index Surface consists of a sphere and a concentric spheroid, with the axis of x as a common axis. As the spheroid may be either oblate or prolate, there should be two classes of crystals in which an ordinary and an extraordinary wave may be propagated. This is found to be the case, and the two types are usually distinguished by the names *positive* and *negative* crystals. The common axis of sphere and spheroid is called the *optic axis* of the crystal, and all crystals of this kind are described as *uniaxal*, to distinguish them from those of a kind dealt with later. It will be observed that in the direction of the optic axis there is only one value of v, so that in this direction there is only one wave propagated, and so only one refracted wave corresponding to an incident one. For all other directions, however, there are two waves propagated, and this phenomenon is spoken of as double refraction.

We have seen that, for the ordinary wave, v and therefore also

μ (the refractive index) is a constant, independent of the direction of propagation. This deduction from theory has been completely verified by experiment, the differences in the estimates of the refractive index from theory and experiment being, with careful experiments, never greater than ·00002, and well within the limits of experimental errors.

For the extraordinary wave, if θ be the angle between the wave normal and the optic axis, we have

$$\cos\theta = l, \text{ so that } v^2 = c^2\cos^2\theta + a^2\sin^2\theta,$$

or

$$\frac{1}{\mu^2} = \frac{\cos^2\theta}{\mu_1^2} + \frac{\sin^2\theta}{\mu_2^2}.$$

Here μ is the refractive index for a wave whose normal is in the direction θ, and μ_1 and μ_2 are the principal refractive indices, i.e. the indices for waves whose normals are parallel and perpendicular respectively to the optic axis. This relation was put to the test of experiment by Stokes and Glazebrook. Stokes found that the difference between theory and experiment in the estimate of μ did not exceed a unit in the fourth place of decimals, and this result was confirmed by Glazebrook, who found that the mean discrepancy in sixty experiments was ·000055, irrespective of sign Glazebrook experimented with different colours (the rays C, F and g) and concluded that in all cases the differences between theory and experiment were comparable with the probable errors of experiment. The following are his results (p. 136) in the case of the ray F.

These results might be represented graphically, but unless a very large scale were used the differences between theory and experiment could not be made visible, since the radii vectores of the two curves corresponding to theory and observation would not differ by more than one ten-thousandth part of either.

Returning to the consideration of the general case where a, b, and c are all different, we can find the equation of the Index Surface by making the same substitutions as before.

We thus get

$$x^2/(1-a^2r^2) + y^2/(1-b^2r^2) + z^2/(1-c^2r^2) = 0.$$

The sections of this surface by the coordinate planes are easily proved to consist of a circle and an ellipse in each case. Parts of these sections are represented in Fig. 51, for the case in which

$a > b > c$. We have seen that, in general, there are two different velocities for each direction of the wave normal, or what is the

θ	μ (theory)	μ (exp.)	Differ-ence	θ	μ (theory)	μ (exp.)	Differ-ence
0° 2′40″	1·66779	1·66780	+·00001	46°46′ 2″	1·56645	1·56653	+·00008
4 19 58	1·66660	1·66663	+·00003	49 23 10	1·55861	1·55876	+·00015
7 51 58	1·66387	1·66385	−·00002	52 42 6	1·54902	1·54914	+·00012
11 23 12	1·65967	1·65978	+·00011	58 39 10	1·53303	1·53312	+·00009
17 8 26	1·64987	1·64996	+·00009	61 39 33	1·52570	1·52573	+·00003
20 26 1	1·64279	1·64287	+·00008	63 9 6	1·52228	1·52241	+·00013
23 50 45	1·63451	1·63455	+·00004	66 14 27	1·51579	1·51571	−·00008
25 49 35	1·62934	1·62930	−·00004	72 18 55	1·50476	1·50475	−·00001
29 18 42	1·61965	1·61974	+·00009	75 36 18	1·50009	1·50005	−·00004
34 48 0	1·60336	1·60336	·00000	79 6 26	1·49612	1·49610	−·00002
35 58 47	1·59048	1·59058	+·00010	80 14 4	1·49507	1·49507	·00000
40 49 21	1·58478	1·58487	+·00009	87 6 40	1·49112	1·49114	+·00002
45 45 57	1·57000	1·57014	+·00014	89 49 6	1·49074	1·49074	·00000

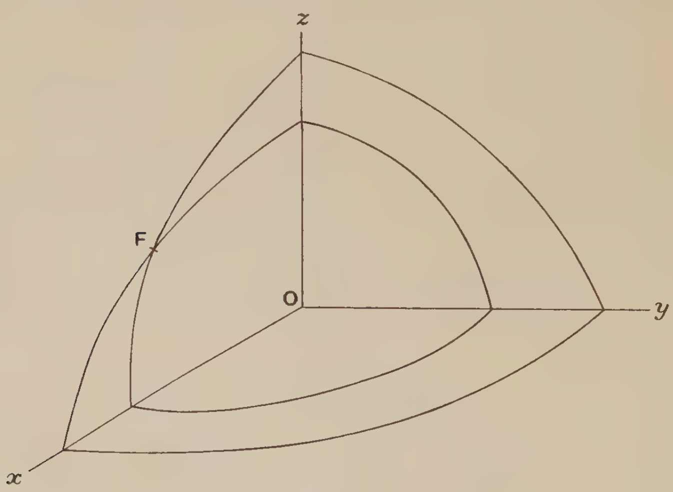

Fig. 51.

same thing, that any radius vector cuts the Index Surface in two points. An inspection of the figure shows, however, that there are exceptions to this general rule. In the plane $y = 0$ the circle and the ellipse intersect in the points F and F', where

$$z/x = \pm \sqrt{(b^2 - c^2)/(a^2 - b^2)};$$

and for each of the directions OF and OF' there is only one index of refraction, and only one wave, and this wave moves with velocity b. Moreover it is easy to prove that OF and OF' are the only two directions in which the two velocities of wave propagation are equal. For solving the equation

$$l^2/(v^2 - a^2) + m^2/(v^2 - b^2) + n^2/(v^2 - c^2) = 0,$$

and putting

$$l^2(b^2 - c^2) = \lambda_1, \quad m^2(a^2 - c^2) = \mu_1, \quad n^2(a^2 - b^2) = \nu_1$$

for brevity, where λ, μ, ν are all positive, we get

$$2v^2 = l^2(b^2 + c^2) + m^2(c^2 + a^2) + n^2(a^2 - b^2) \pm \sqrt{(\lambda_1 - \mu_1 - \nu_1)^2 + 4\lambda_1\mu_1}.$$

The condition for equal velocities is

$$(\lambda_1 - \mu_1 - \nu_1)^2 + 4\lambda_1\mu_1 = 0,$$

and since λ_1 and μ_1 are positive this requires

$$\lambda_1\mu_1 = 0, \text{ and } \lambda_1 - \mu_1 - \nu_1 = 0,$$

so that we must have $\mu_1 = 0$, and $\lambda_1 = \nu_1$, i.e. $m = 0$, and

$$n/l = \pm \sqrt{(b^2 - c^2)/(a^2 - b^2)}.$$

The two lines OF and OF' are called the *optic axes* of the crystal, or the axes of single wave velocity, and crystals with two such axes are styled *biaxal* crystals. In Fig. 51 the axis of x is the bisector of the acute angle between the optic axes, so that in this case this bisector coincides with the least axis of the ellipsoid

$$a^2x^2 + b^2y^2 + c^2z^2 = 1.$$

In such circumstances the crystal is said to be *negative*, while if the bisector coincides with the greatest axis the crystal is called *positive*. If ω be the angle between the optic axes we have

$$\tan \omega/2 = \tan FOx = \sqrt{(b^2 - c^2)/(a^2 - b^2)},$$

so that the crystal is positive or negative as $(b^2 - c^2)/(a^2 - b^2)$ is

greater or less than unity, i.e. according as $2b^2$ is greater or less than a^2+c^2. This formula for the angle between the optic axes can be tested by experiment, for the optical constants a, b, c as well as the angle ω can all be determined experimentally. The constants a, b, c vary with the colour of the light employed, so that the angle between the optic axes varies also with the colour, a fact known as the dispersion of the optic axes. The following table gives the value of ω calculated from the above formula, and compared with its measure obtained experimentally in the case of aragonite for different rays of the spectrum.

Ray	ω (observed)	ω (theory)	Difference	Percentage difference
B (red)	18° 5′	17° 58′	7′	0·6
E	18° 17′	18° 3′	14′	1·3
H (violet)	18° 40′	18° 27′	13′	1·2

A very slight error in the estimate of the optical constants would account for the small differences between observation and theory. The test is a delicate one, and the agreement satisfactory.

The experimental investigation of the form of the index surface has been undertaken by Glazebrook, and he has compared

θ	μ_1 (theory)	μ_1 (exp.)	Difference	θ	μ_2 (theory)	μ_2 (exp.)	Difference
0	1·68123	1·68119	− ·00004	0	1·68506	1·68500	− ·00006
2° 9′14″	1·68124	1·68125	+ ·00001	3°11′24″	1·68578	1·68578	·00000
4 14 15	1·68125	1·68130	+ ·00005	5 14 5	1·68567	1·68568	+ ·00001
7 13 27	1·68123	1·68119	− ·00004	7 12 12	1·68514	1·68517	+ ·00003
9 59 43	1·68118	1·68123	+ ·00005	9 59 26	1·68371	1·68376	+ ·00005
12 7 8	1·68096	1·68104	+ ·00008	11 43 33	1·68251	1·68226	− ·00025
14 47 18	1·67893	1·67918	+ ·00025	14 4 34	1·68152	1·68122	− ·00030
17 18 49	1·67574	1·67591	+ ·00017	17 44 11	1·68137	1·68134	− ·00003
19 29 6	1·67247	1·67274	+ ·00027	18 59 25	1·68136	1·68127	− ·00009

the results of theory and experiment for a number of arcs taken from widely different portions of the surface. The table (p. 138) gives some of his results, μ_1 and μ_2 being the refractive indices of the inner and outer sheets of the index surface, and θ the angle that the wave normal makes with a fixed line in the plane of the section under discussion.

The greatest difference is less than 1 in 6000, so that the discrepancy between theory and observation would not be made manifest to the eye by drawing, unless a very large scale were used. The average difference is very much less than this, and is not far from the limit of experimental error, while most of the differences are within that limit.

Another arc examined by Glazebrook gave the following results:

θ	μ_1 (theory)	μ_1 (exp.)	Difference	θ	μ_2 (theory)	μ_2 (exp.)	Difference
0	1·53014	1·53014	·00000	0	1·68242	1·68218	−·00024
3°26′53″	1·53016	1·53010	−·00006	3° 0′40″	1·68263	1·68254	−·00009
7 54 17	1·53019	1·53013	−·00006	6 56 11	1·68292	1·68289	−·00003
17 43 19	1·53028	1·53028	·00000	10 18 37	1·68317	1·68336	+·00019
22 45 50	1·53035	1·53036	+·00001	13 8 42	1·68339	1·68363	+·00024
26 58 44	1·53043	1·53045	+·00002	18 45 47	1·68383	1·68421	+·00038

For the inner sheet the agreement is very close, the differences being all within the limits of experimental error. The average difference, irrespective of sign, is about one in 60,000, and the greatest difference one in 25,000. For the outer sheet the agreement is less satisfactory, the average difference irrespective of sign being about one in 9000, and the greatest difference one in 4000.

A third arc gave the following results:

θ	μ_1 (theory)	μ_1 (exp.)	Difference	θ	μ_2 (theory)	μ_2 (exp.)	Difference
0	1·68103	1·68099	− ·00004	0	1·68533	1·68526	− ·00007
3°12′50″	1·67714	1·67721	+ ·00007	7° 9′10″	1·68465	1·68454	− ·00011
13 6 20	1·66298	1·66300	+ ·00002	17 2 40	1·68445	1·68448	+ ·00003
21 4 30	1·64607	1·64603	− ·00004	25 0 50	1·68443	1·68452	+ ·00009
28 14 10	1·62824	1·62807	− ·00017	32 10 30	1·68443	1·68447	+ ·00004
35 29 20	1·60900	1·60897	− ·00003	38 27 30	1·68444	1·68453	+ ·00009
45 14 50	1·58363	1·58365	+ ·00002	49 13 0	1·68445	1·68457	+ ·00012
60 1 30	1·55154	1·58157	+ ·00003	63 59 30	1·68447	1·68452	+ ·00005
69 37 40	1·53784	1·53774	− ·00010	73 35 50	1·68448	1·68444	− ·00004

In this case we have a large arc over which, for both the inner and the outer sheets of the index surface, the agreement between theory and experiment is remarkably close. In view of all the results given, there can be no doubt that the theory represents the facts, at any rate as an excellent first approximation, which is all that need be claimed for it.

Feeling sure that we are on solid ground, in so far as this may be tested by experiments such as those just referred to, we may proceed to further developments of the theory.

(1) The equations (II) and (VI) above (p. 132) are those obtained in Solid Geometry when investigating the direction cosines (λ, μ, ν) of the principal axes of a central section of the ellipsoid

$$a^2x^2 + b^2y^2 + c^2z^2 = 1$$

made by the plane

$$lx + my + nz = 0.$$

We see then that the curl of the displacement is in the direction of the principal axes of this section.

(2) It follows from (1) that the two waves propagated in any given direction are plane polarised, their planes of polarisation being at right angles. The use of doubly refracting media is, in fact, one of the most usual means of obtaining plane polarised light.

(3) In the case of waves moving in the direction of an optic axis the two velocities are the same, so that the principal axes of the section of the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ by the wave front $lx + my + nz = 0$ are equal. The section is therefore a circle, and the optic axes are the normals to the circular sections of the ellipsoid.

(4) Let $R'PRQ$ be a section of the ellipsoid by the wave front, ON the wave normal, OF and OF' the optic axes. If OR and

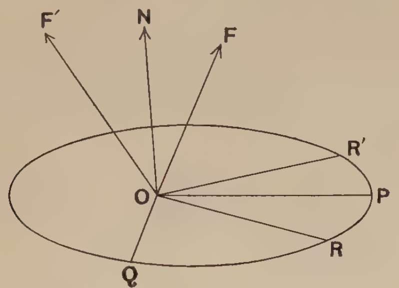

Fig. 52.

OR' be sections of the wave front by the circular sections normal to OF and OF' respectively, then we have $OR = 1/b = OR'$. As the principal axes of an ellipse bisect the angles between a pair of equal radii vectores, we see from (1) that the directions of the curls are the bisectors of the angles between OR and OR'.

(5) Let ψ and ψ' be the angles between the optic axes and the curl OP, θ and θ' the angles between the wave normal ON and the optic axes OF and OF', 2χ the angle between the planes ONF and ONF'.

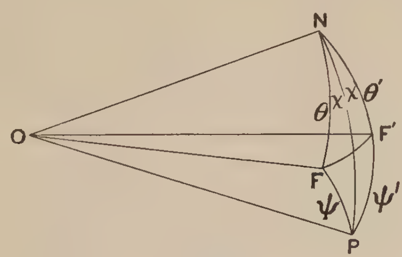

Fig. 53.

It follows from (4) that $PNF = PNF' = \chi$.

The direction cosines of the optic axes are

$$\sqrt{(a^2-b^2)/(a^2-c^2)}, \quad 0, \quad \pm\sqrt{(b^2-c^2)/(a^2-c^2)}.$$

Hence we have, from (V) p. 132,

$$v^2 = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2 = b^2 + \lambda^2(a^2-b^2) - \nu^2(b^2-c^2)$$

$$= b^2 + (a^2-c^2)\left[\lambda\sqrt{(a^2-b^2)/(a^2-c^2)} + \nu\sqrt{(b^2-c^2)/(a^2-c^2)}\right]$$
$$\left[\lambda\sqrt{(a^2-b^2)/(a^2-c^2)} - \nu\sqrt{(b^2-c^2)/(a^2-c^2)}\right]$$

$$= b^2 + (a^2-c^2)\cos\psi\cos\psi'.$$

Let v_1 and v_2 be the two velocities of waves travelling in the direction ON, and ψ_1, ψ_2, ψ_1', ψ_2' be the corresponding angles as defined above. We then have

$$v_1^2 + v_2^2 = 2b^2 + (a^2-c^2)(\cos\psi_1\cos\psi_1' + \cos\psi_2\cos\psi_2'),$$

and $$v_1^2 - v_2^2 = (a^2-c^2)(\cos\psi_1\cos\psi_1' - \cos\psi_2\cos\psi_2').$$

From Fig. 53 we see that

$$\cos\psi_1 = \sin\theta\cos\chi, \qquad \cos\psi_1' = \sin\theta'\cos\chi,$$

$$\cos\psi_2 = \sin\theta\sin\chi, \text{ and } \cos\psi_2' = -\sin\theta'\sin\chi.$$

Making use of these relations we get

$$v_1^2 - v_2^2 = (a^2-c^2)\sin\theta\sin\theta',$$

and $$v_1^2 + v_2^2 = 2b^2 + (a^2-c^2)\sin\theta\sin\theta'\cos 2\chi.$$

The first of these relations represents a law first suggested by Biot as a generalisation from his experimental results.

We also have

$$\cos FF' = (a^2-b^2)/(a^2-c^2) - (b^2-c^2)/(a^2-c^2)$$

$$= (a^2-2b^2+c^2)/(a^2-c^2) = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos 2\chi.$$

$$\therefore\ v_1^2 + v_2^2 = a^2 + c^2 - (a^2-c^2)\cos\theta\cos\theta'.$$

Thus finally we get

$$2v_1^2 = a^2 + c^2 - (a^2-c^2)\cos(\theta+\theta'),$$

and $$2v_2^2 = a^2 + c^2 - (a^2-c^2)\cos(\theta-\theta'),$$

or $$v^2 = c^2 + (a^2-c^2)\sin^2(\theta\pm\theta')/2,$$

which gives the velocities and refractive indices in a form convenient for comparison with experiment.

(6) The results of the last section can also be very simply obtained by the aid of the elementary geometry of sphero-conics. If we regard v as a constant, and (l, m, n) as the coordinates of a point, then the equations

$$l^2 + m^2 + n^2 = 1,$$

and $$l^2/(v^2 - a^2) + m^2/(v^2 - b^2) + n^2/(v^2 - c^2) = 0$$

represent a sphere and a cone respectively, and their intersection a spherical ellipse. Of this ellipse F and F' are the foci, C, the middle point of FF', the centre, and $\theta + \theta'$ the axis major. As the sum of the focal distances is constant for all points of the ellipse, we have $(\theta + \theta')/2 = CN$, when N is in the plane containing FF', i.e. in the plane $y = 0$. Hence putting $m = 0$ we get

$$l^2/(v^2 - a^2) + n^2/(v^2 - c^2) = 0,$$

or $$(v^2 - a^2)/(c^2 - v^2) = l^2/n^2 = \tan^2 (\theta + \theta')/2,$$

so that $$2v^2 = a^2 + c^2 - (a^2 - c^2) \cos (\theta + \theta').$$

The other value of v^2 is obtained by interchanging a and c, and replacing θ' by $\pi - \theta'$. A third method of obtaining these results will be employed when dealing with ray velocities (§ 17 below).

(7) A geometrical construction for the direction of the curl is given in (1) above. If we wish to obtain an analytical expression for this when the velocity (v) of the wave, and the direction (l, m, n) in which it is travelling are given, we have this in the fundamental relation (III, p. 132),

$$\lambda (a^2 - v^2)/l = \mu (b^2 - v^2)/m = \nu (c^2 - v^2)/n,$$

which determines λ, μ, ν.

So far we have been occupied entirely with the consideration of *waves* propagated in a given direction, and we have seen that the index surface lends itself very readily to the discussion of such problems. If, however, we wish to introduce the idea of *rays*, it is more convenient to employ another surface intimately related to the index surface. This is the *Wave Surface* described in Chapter II. It is the envelope of all positions of the wave front at any time. If Q be a point on this surface and ON the perpendicular from the origin on the tangent plane at Q, then ON is the wave normal for the wave corresponding to the point Q, and

$ON = v$. A point P being taken on ON such that $OP \, . \, ON = 1$ (taking the velocity in free ether as unity), we have $OP = 1/v$, so that P is a point on the index surface as defined above. Thus

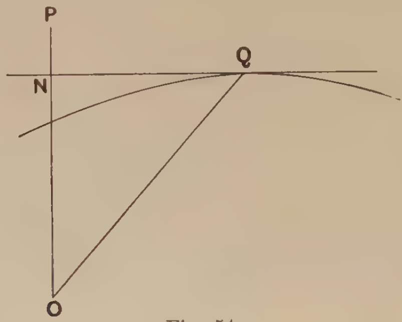

Fig. 54.

the wave surface is the polar reciprocal of the index surface with respect to a unit sphere whose centre is O. Its equation is easily obtained by the methods usually employed for finding the polar reciprocal of a given surface. We have to find the envelope of

$$lx + my + nz = v,$$

subject to the relations

$$l^2 + m^2 + n^2 = 1,$$

and $$l^2/(v^2 - a^2) + m^2/(v^2 - b^2) + n^2/(v^2 - c^2) = 0.$$

Differentiating these equations we get

$$xdl + ydm + zdn - dv = 0, \quad ldl + mdm + ndn = 0,$$

$$ldl/(v^2 - a^2) + mdm/(v^2 - b^2) + ndn/(v^2 - c^2) - vdv\,[l^2/(v^2 - a^2)^2 + m^2/(v^2 - b^2)^2 + n^2/(v^2 - c^2)^2] = 0.$$

From these, using undetermined multipliers, we obtain

$$x + Al + Bl/(v^2 - a^2) = 0, \quad y + Am + Bm/(v^2 - b^2) = 0,$$

$$z + An + Bn/(v^2 - c^2) = 0,$$

and $$1 + Bv\,[l^2/(v^2 - a^2)^2 + m^2/(v^2 - b^2)^2 + n^2/(v^2 - c^2)^2] = 0.$$

Multiplying the first of these equations by l, the second by m, the third by n, and adding we get $v + A = 0$. Transposing the third terms of these same equations, squaring and adding, and putting $x^2 + y^2 + z^2 = r^2$, we get

$$r^2 + 2Av + A^2 = B^2\,[l^2/(v^2 - a^2)^2 + m^2/(v^2 - b^2)^2 + n^2/(v^2 - c^2)^2],$$

so that $$r^2 - v^2 = -B/v.$$

Hence $x = -Al' - Bl/(v^2 - a^2) = lv + lv\,(r^2 - v^2)/(v^2 - a^2)$

$$= lv\,.\,(r^2 - a^2)/(v^2 - a^2),$$

and similarly for y and z.

Thus we have

$$lv = x\,.\,(v^2 - a^2)/(r^2 - a^2) = x + x\,(v^2 - r^2)/(r^2 - a^2),$$

and so for mv and nv. Hence

$$\begin{aligned} v^2 &= v\,(lx + my + nz) \\ &= r^2 + (v^2 - r^2)\,[x^2/(r^2 - a^2) + y^2/(r^2 - b^2) + z^2/(r^2 - c^2)] \end{aligned}$$

or $$x^2/(r^2 - a^2) + y^2/(r^2 - b^2) + z^2/(r^2 - c^2) = 1,$$

which is the equation of the wave surface*.

The geometrical peculiarities of the wave surface have been very thoroughly investigated, having attracted the attention of a number of mathematicians of the first rank. It would, however, be outside the purpose of this book to enter at any length into this subject. We are concerned with the wave surface only so far as its properties help us to understand the optical behaviour of crystals, and for this end only the elements of the geometry of the surface are required.

(8) We have seen that the wave surface is the polar reciprocal of the index surface, and from this fact or from the equation just

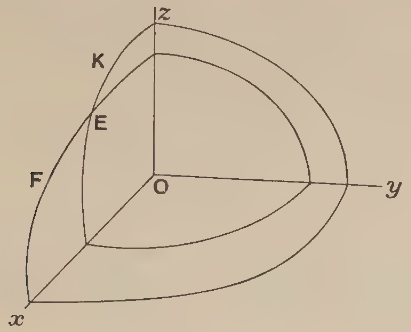

Fig. 55.

obtained, the general form of the surface can be readily imagined. It consists of two sheets symmetrical about the coordinate planes.

* This equation can also be readily obtained by means of a few elegant theorems as to apsidal surfaces due to MacCullagh, and contained in the first paper of his *Collected Works*, and also in Salmon's *Geometry of Three Dimensions*, Chap. XIV.

The sections of the surface by these planes are ellipses and circles, which intersect in real points E and E' in the plane $y=0$, when $a>b>c$. The equations to the lines joining O to these points E and E' are

$$z/x = \pm (a/c) \,.\, \sqrt{(b^2-c^2)/(a^2-b^2)}.$$

(9) The refracted rays corresponding to any incident ray are, as explained in Chap. II., given by Huyghens' construction with the aid of the wave surface. Suppose that a ray OA is incident

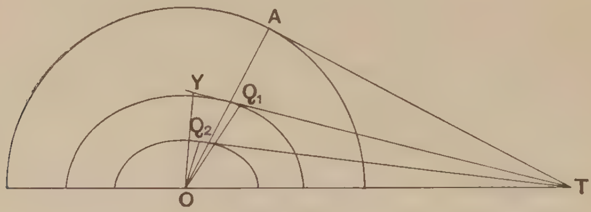

Fig. 56.

on a given plane which cuts the plane of the paper in OT. Let this ray meet the sphere whose radius is V (the velocity of light in the external medium) in A, then the tangent plane at A to this sphere cuts the given plane in a line through T at right angles to the plane of the paper. If through this line we draw tangent planes to the wave surface whose centre is O, these planes will touch the surface in two points Q_1 and Q_2, and OQ_1 and OQ_2 are the directions of the two refracted rays.

(10) From equation (III) p. 132, we have

$$l/\lambda = (a^2-v^2)/\kappa,$$

where
$$\kappa = a^2 l\lambda + b^2 m\mu + c^2 n\nu.$$

Also if L_1, M_1, N_1 be the direction cosines of the ray corresponding to the point (x, y, z) on the wave surface we have, by means of the equations obtained above when finding the equation of the wave surface,

$$L_1 r = x = lv - lv\,(r^2-v^2)/(a^2-v^2) = lv - v\,(r^2-v^2)\,\lambda/\kappa.$$

By elimination from this and the two similar equations, we get

$$\begin{vmatrix} L_1 & l & \lambda \\ M_1 & m & \mu \\ N_1 & n & \nu \end{vmatrix} = 0,$$

which proves that the ray, the wave normal, and the curl are coplanar.

(11) The direction of the curl in the wave corresponding to the ray OQ is easily obtained. If N be the foot of the perpendicular from O on the wave front, its coordinates are lv, mv, and nv. Also from the last section (10) we have

$$lv - x = v(r^2 - v^2)\lambda/\kappa,$$

so that the direction cosines of NQ, which are proportional to $lv - x$, $mv - y$, $nv - z$, are proportional to λ, μ, ν. Hence NQ is the direction of the curl, and the curl is therefore directed along the projection of the ray on the wave front.

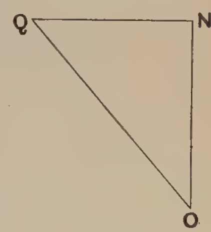

Fig. 57.

We see also from the above that

$$NQ = v(r^2 - v^2)/\kappa,$$

whence, since $\quad NQ = \sqrt{OQ^2 - ON^2} = \sqrt{r^2 - v^2},$

we derive the relations

$$NQ = \sqrt{r^2 - v^2} = \kappa/v.$$

(12) A useful formula for κ^2 may be obtained as follows:

We have $\quad l^2/(a^2 - v^2) + m^2/(b^2 - v^2) + n^2/(c^2 - v^2) = 0.$

Putting $a^2 - v^2 = w$ we get a quadratic equation for w of the form

$$w^2 + Aw + l^2(a^2 - b^2)(a^2 - c^2) = 0.$$

Hence if w_1 and w_2 be the roots of this equation, we have

$$l^2 = w_1 w_2/(a^2 - b^2)(a^2 - c^2) = (a^2 - v_1^2)(a^2 - v_2^2)/(a^2 - b^2)(a^2 - c^2).$$

Similar expressions for m^2 and n^2 may be written down from symmetry.

Also we have

$$(a^2 - v^2)\lambda = \kappa l, \quad (b^2 - v^2)\mu = \kappa m, \quad (c^2 - v^2)\nu = \kappa n.$$

Thus

$$\begin{aligned} 1/\kappa^2 &= \lambda^2/\kappa^2 + \mu^2/\kappa^2 + \nu^2/\kappa^2 \\ &= [l/(a^2 - v^2)]^2 + [m/(b^2 - v^2)]^2 + [n/(c^2 - v^2)]^2, \end{aligned}$$

so that we get

$$\frac{1}{\kappa_1^2} = \frac{a^2 - v_2^2}{(a^2 - b^2)(a^2 - c^2)(a^2 - v_1^2)} + \frac{b^2 - v_2^2}{(b^2 - c^2)(b^2 - a^2)(b^2 - v_1^2)} + \frac{c^2 - v_2^2}{(c^2 - a^2)(c^2 - b^2)(c^2 - v_1^2)}$$

$$= \frac{v_1^2 - v_2^2}{(a^2 - v_1^2)(b^2 - v_1^2)(c^2 - v_1^2)},$$

and

$$\frac{1}{\kappa_2^2} = \frac{v_2^2 - v_1^2}{(a^2 - v_2^2)(b^2 - v_2^2)(c^2 - v_2^2)}.$$

These formulae for κ_1 and κ_2 can be put in another form by means of the expressions for v_1 and v_2 obtained on p. 142. From these we have

$$b^2 - v_1^2 = -(a^2 - c^2)\cos\psi_1 \cos\psi_1' = -(a^2 - c^2)\sin\theta \sin\theta' \cos^2\chi,$$

$$a^2 - v_1^2 = \quad (a^2 - c^2)\cos^2(\theta + \theta')/2,$$

and

$$c^2 - v_1^2 = -(a^2 - c^2)\sin^2(\theta + \theta')/2.$$

Whence we get

$$2\kappa_1 = \pm(a^2 - c^2)\sin(\theta + \theta')\cos\chi,$$

and

$$2\kappa_2 = \pm(a^2 - c^2)\sin(\theta - \theta')\sin\chi.$$

(13) The angle between the ray and the wave normal is NOQ and we have

$$\tan NOQ = NQ/ON = \kappa/v^2.$$

Hence

$$\tan NOQ_1 = \kappa_1/v_1^2 = \pm\frac{a^2 - c^2}{2v_1^2}\sin(\theta + \theta')\cos\chi$$

$$= \pm\frac{(a^2 - c^2)\sin(\theta + \theta')\cos\chi}{a^2 + c^2 - (a^2 - c^2)\cos(\theta + \theta')}$$

and

$$\tan NOQ_2 = \pm\frac{(a^2 - c^2)\sin(\theta - \theta')\sin\chi}{a^2 + c^2 - (a^2 - c^2)\cos(\theta - \theta')}.$$

It should be noted that if N coincides with F or F' (Fig. 53), the angle χ is indeterminate, and these formulae become indeterminable. Also in the special case of a uniaxal crystal we have $\chi = 0$, and $\theta = \theta'$, so that $NOQ_2 = 0$, and

$$\tan NOQ_1 = \frac{(a^2 - c^2)\sin\theta\cos\theta}{v_1^2}.$$

(14) We have seen (p. 137) that, in general, any radius vector to the index surface cuts that surface in two points. Reciprocating we see that in general there are two parallel tangent planes to the wave surface for each direction, so that there are two *rays* corresponding to a given wave normal, the directions of these two rays being determined by the equations in the last section (13). However, in the direction of an optic axis, such as OF, there is only one radius vector to the index surface, whence we see, by reciprocation, that for a wave normal in this direction there is only one tangent plane to the wave surface. On investigation it appears that this plane touches the surface not in a single point, but in an infinity of points forming a circle. Hence corresponding to a wave normal along the optic axis there are not two rays, but an infinity of rays joining the origin to the various points of this circle. This, which was first predicted from theory*, has been verified by experiment, and is known as the phenomenon of internal conical refraction.

From the investigation on p. 144 it appears that the coordinates of a point on the wave surface may be written in the form

$$x = lv\,(r^2 - a^2)/(v^2 - a^2), \quad y = mv\,(r^2 - b^2)/(v^2 - b^2),$$
$$z = nv\,(r^2 - c^2)/(v^2 - c^2).$$

At the extremity of an optic axis we have $y = 0$ and $v = b$, so that

$$x/(r^2 - a^2) = lb/(b^2 - a^2) = -\,b/\sqrt{(a^2 - b^2)(a^2 - c^2)},$$

and $$z/(r^2 - c^2) = nb/(b^2 - c^2) = \quad b/\sqrt{(b^2 - c^2)(a^2 - c^2)}.$$

Hence the extremities of the rays, corresponding to a wave normal along the optic axis, lie on the intersection of the spheres

$$r^2 - a^2 + x\sqrt{(a^2 - b^2)(a^2 - c^2)}/b = 0,$$

and $$r^2 - c^2 - z\sqrt{(b^2 - c^2)(a^2 - c^2)}/b = 0.$$

The intersection of these spheres is a circle in the plane $\dfrac{x}{b}\sqrt{\dfrac{a^2 - b^2}{a^2 - c^2}} + \dfrac{z}{b}\sqrt{\dfrac{b^2 - c^2}{a^2 - c^2}} = 1.$ Thus the wave front touches the surface at all the points of this circle. The cone of rays is formed of lines from the origin to the points of intersection of the above spheres and plane. The equation of this cone is obtained by

* By Sir William Hamilton.

forming a homogeneous equation of the second degree by the aid of the equations of these surfaces. In this way we obtain the equation

$$r^2 + \frac{x\sqrt{a^2-b^2}}{b^2}(x\sqrt{a^2-b^2} + z\sqrt{b^2-c^2})$$
$$= \frac{a^2}{b^2(a^2-c^2)}[x\sqrt{a^2-b^2} + z\sqrt{b^2-c^2}]^2,$$

i.e. $$a^2(b^2-c^2)x^2 + b^2(a^2-c^2)y^2 + c^2(a^2-b^2)z^2$$
$$= xz(a^2+c^2)\sqrt{(a^2-b^2)(b^2-c^2)}.$$

By considering the section of this cone by the plane of symmetry $y=0$, we readily find that, if Φ be the vertical angle of the cone, $\tan\Phi = \sqrt{(b^2-c^2)(a^2-b^2)}/b^2$.

This result can also be obtained from an examination of a section of the wave surface by the plane $y=0$. This section consists of the circle $x^2+z^2=b^2$, and the ellipse $x^2/b^2+z^2/a^2=1$. If $x\cos\theta_1 + z\sin\theta_1 = b$ be the common tangent to these two curves, we have $b^2 = c^2\cos^2\theta_1 + a^2\sin^2\theta_1$, and therefore

$$\tan\theta_1 = \sqrt{(b^2-c^2)/(a^2-b^2)}.$$

The coordinates of the point of contact of this tangent with the ellipse are $x=(c^2/b)\cos\theta_1$ and $z=(a^2/b)\sin\theta_1$, so that if θ_2 be the vectorial angle to the point of contact, we have

$$\tan\theta_2 = z/x = (a^2/c)\tan\theta_1 = (a^2/c^2)\sqrt{(b^2-c^2)/(a^2-b^2)}.$$

Hence $\tan\Phi = \tan(\theta_2-\theta_1) = \sqrt{(b^2-c^2)(a^2-b^2)}/b^2$ as before. The angle Φ can be measured experimentally, and the results obtained are found to agree closely with those calculated from the formula just obtained. In the original experiments made by Lloyd with aragonite the angle of the cone was found to be 1° 50′. Its magnitude as indicated by the above theory was 1° 55′, the difference being within the limits of the experimental errors.

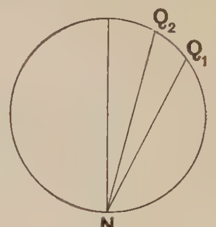

Fig. 58.

The direction of the curl has been proved in § 11 above to be along the projection of the ray on the wave front. Hence the various rays forming the cone of internal conical

refraction are polarised in different azimuths NQ_1, NQ_2 ... (Fig. 58); so that if the incident light were plane polarised, there would only be one refracted ray. The polarisation of the different rays is found experimentally to be that predicted by the above theory.

(15) The wave surface represents the confines of disturbances that have travelled from the origin in a given time, so that its radii vectores are proportional to the ray velocities. As the surface consists of two sheets there are, in general, two ray velocities for a given direction of the ray. An exception occurs, however, for the rays OE and OE' (Fig. 55), the points E and E' being singular points on the surface. Corresponding to each of the rays OE and OE' there is only one ray velocity, and these lines are consequently called the *axes of single ray velocity*, or more briefly the *ray axes*. The angle EOF between a ray axis and the corresponding optic axis is obtained from the formulae on pp. 137 and 146. We have

$$\tan EOF = \tan(EOx - FOx) = \frac{(a/c - 1)\sqrt{(b^2 - c^2)/(a^2 - b^2)}}{1 + (a/c)(b^2 - c^2)/(a^2 - b^2)}$$
$$= \frac{\sqrt{(a^2 - b^2)(b^2 - c^2)}}{b^2 + ac}.$$

(16) Formulae for the ray velocities corresponding to those for the wave velocities given on p. 142 can be found by methods similar to those employed there. The equation of the wave surface is $a^2x^2/(r^2 - a^2) + b^2y^2/(r^2 - b^2) + c^2z^2/(r^2 - c^2) = 0$, where $r^2 = x^2 + y^2 + z^2$. If we regard r as constant, these two equations represent a cone and a sphere, and their intersection is a spherical ellipse of which E and E' are the foci. If ϕ and ϕ' be the angles that a ray makes with the axes of single ray velocity, then, by the properties of a spherical ellipse, $\phi + \phi'$ is constant and equal to the axis major. By taking the special case when $y = 0$, we get

$$a^2x^2/(r^2 - a^2) + c^2z^2/(r^2 - c^2) = 0,$$

whence
$$\frac{r^2 - a^2}{c^2 - r^2} = \frac{a^2x^2}{c^2z^2} = \frac{a^2}{c^2}\tan^2\frac{\phi + \phi'}{2}.$$

This gives $2/r^2 = 1/a^2 + 1/c^2 + (1/a^2 - 1/c^2)\cos(\phi + \phi')$, and the other value of $2/r^2$ is obtained by interchanging a and c, and

replacing ϕ' by $\pi - \phi'$. Also since the normal to a spherical ellipse bisects the angle between the focal radii, we see that the

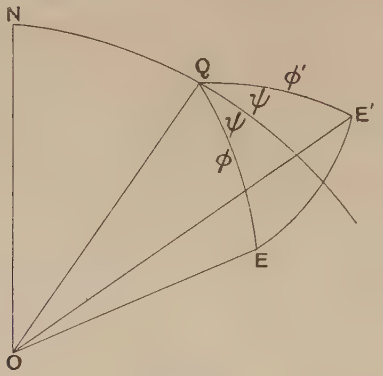

Fig. 59.

curls corresponding to the two rays in a given direction OQ lie in planes that bisect the angles between the planes through OQ and the ray axes OE and OE' (Fig. 59).

(17) The formula for r in the last paragraph can also be obtained without the aid of the geometry of sphero-conics. If L_1, M_1, N_1 be the direction cosines of the ray we have, from the equation of the wave surface,

$$a^2L_1^2/(r^2-a^2) + b^2M_1^2/(r^2-b^2) + c^2N_1^2/(r^2-c^2) = 0,$$

and if r_1^2 and r_2^2 are the roots of this equation

$$1/r_1^2 + 1/r_2^2 = \Sigma\,(1/b^2 + 1/c^2)\,L_1^2, \text{ and } 1/r_1^2r_2^2 = \Sigma\,(L_1^2/b^2c^2).$$

The direction cosines of the ray axes are

$$(c/b)\sqrt{(a^2-b^2)/(a^2-c^2)},\ 0,\ \pm (a/b)\sqrt{(b^2-c^2)/(a^2-c^2)}.$$

Hence we have

$$\cos\phi = L_1\,(c/b)\sqrt{(a^2-b^2)/(a^2-c^2)} + N_1\,(a/b)\sqrt{(b^2-c^2)/(a^2-c^2)},$$

and

$$\cos\phi' = L_1\,(c/b)\sqrt{(a^2-b^2)/(a^2-c^2)} - N_1\,(a/b)\sqrt{(b^2-c^2)/(a^2-c^2)}.$$

Thus
$$L_1 = \frac{\cos\phi + \cos\phi'}{2}\,\frac{b}{c}\sqrt{\frac{a^2-c^2}{a^2-b^2}},$$

and
$$N_1 = \frac{\cos\phi - \cos\phi'}{2}\,\frac{b}{a}\sqrt{\frac{a^2-c^2}{b^2-c^2}}.$$

By the aid of these relations we get

$$\begin{aligned} 1/r_1^2 + 1/r_2^2 &= (1/b^2 + 1/c^2)\, L_1^2 + (1/c^2 + 1/a^2)\, M_1^2 + (1/a^2 + 1/b^2)\, N_1^2 \\ &= (1/b^2 + 1/c^2)\, L_1^2 + (1/c^2 + 1/a^2)(1 - L_1^2 - N_1^2) \\ &\qquad\qquad + (1/a^2 + 1/b^2)\, N_1^2 \\ &= 1/a^2 + 1/c^2 + (1/a^2 - 1/c^2) \cos\phi \cos\phi', \end{aligned}$$

and $$\begin{aligned} 4/r_1^2 r_2^2 = 4/a^2c^2 &+ (1/a^2 - 1/c^2)^2 (\cos^2\phi + \cos^2\phi') \\ &+ 2\,(1/a^4 - 1/c^4) \cos\phi \cos\phi', \end{aligned}$$

so that $1/r_1^2 - 1/r_2^2 = (1/a^2 - 1/c^2) \sin\phi \sin\phi'$, which in conjunction with the formula for $1/r_1^2 + 1/r_2^2$ yields the same results as before.

(18) The direction of a ray on emerging from a crystal is determined by the position of the tangent plane to the wave surface at the point where the ray meets that surface. For an ordinary point there is only one tangent plane, so that the direction of the emerging ray is quite definite. The points E and E', however, are singular points on the wave surface. At such a point there is not a single tangent plane and a single normal, but an infinity of tangent planes enveloping a tangent cone and an infinity of normals forming a normal cone. It follows, then, that to a single ray within the crystal in the direction of a ray axis, there corresponds a cone of emerging rays, each ray having a different plane of polarisation. This prediction from theory has been verified by experiment, and the phenomenon thus described is known as external conical refraction.

To investigate the form of the normal cone we have to make use of the formulae for the coordinates of any point of the wave surface,

$$x = lv\,(r^2 - a^2)/(v^2 - a^2), \qquad y = mv\,(r^2 - b^2)/(v^2 - b^2),$$

and $$z = nv\,(r^2 - c^2)/(v^2 - c^2).$$

At E a point on the ray axis we have $y = 0$, and

$$x = c\sqrt{(a^2 - b^2)/(a^2 - c^2)}, \qquad z = a\sqrt{(b^2 - c^2)/(a^2 - c^2)}.$$

Substituting these in the above equations for x and z, we get

$$v^2 - a^2 + (lv/c)\sqrt{(a^2 - b^2)/(a^2 - c^2)} = 0,$$

and $$v^2 - c^2 - (nv/a)\sqrt{(b^2 - c^2)/(a^2 - c^2)} = 0.$$

On eliminating v from these two equations we obtain

$$l^2(a^2 - b^2) + n^2(b^2 - c^2) + ln\sqrt{(a^2 - b^2)(b^2 - c^2)}\,(a^2 + c^2)/ac = a^2 - c^2.$$

In this equation l, m, n are the direction cosines of any normal through E, so that the equation of the normal cone referred to E as origin is

$$x^2(b^2-c^2)+y^2(a^2-c^2)+z^2(a^2-b^2) = xz\sqrt{(a^2-b^2)(b^2-c^2)}(a^2+c^2)/ac.$$

By considering the section of this cone by the plane of symmetry ($y=0$), we find that the angle Ψ of the cone is given by the formula $\tan\Psi=\sqrt{(b^2-c^2)(a^2-b^2)}/ac$. This can also be obtained by an examination of the section of the wave surface by the plane $y=0$. The section consists of a circle and an ellipse, and the angle of the cone is the angle between the normals to these two curves at their point of intersection.

(19) The normal cone discussed in the last paragraph is not the same as the cone of rays that emerge from the crystal. This latter cone is obtained by means of Huyghens' construction and, by way of illustration, we shall use this construction to find the section of the cone of rays by the plane of symmetry. The section of the wave surface in the crystal by this plane of symmetry consists, as has been seen, of a circle and an ellipse intersecting at E (Fig. 60).

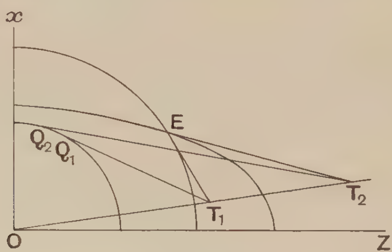

Fig. 60.

Let OT_1T_2 be the section of the surface of the crystal by the plane $y=0$, and let the tangents at E to the circle and the ellipse meet this line in T_1 and T_2. From these points tangents T_1Q_1 and T_2Q_2 are to be drawn to the circle of radius V, whose centre is at O, this circle being the section of the spherical wave surface for the waves in the external medium, where the velocity of propagation is V. Then, by the principles explained when describing

Huyghens' construction, OQ_1 and OQ_2 are the emergent rays in the plane under consideration, and as this plane is the plane of symmetry the angle between these lines is the angle of the cone of external conical refraction. It is thus a matter of elementary geometry to calculate the angle of this cone for any given circumstances. Thus when the surface of the crystal is perpendicular to the ray axis OE, we find the angle given by

$$\sin \chi = V\sqrt{(a^2 - b^2)(b^2 - c^2)}/abc.$$

The polarisation of the different emergent rays is found experimentally to agree with the theory, and there is also a good agreement with reference to the angle of the cone of external conical refraction. The mean of Lloyd's measurements on aragonite gave $\chi = 2^\circ\, 59'$, and the calculated value was $3^\circ\, 1'$, the difference being within the limits of the errors of experiment.

(20) Before closing this chapter we shall investigate the direction of the flow of energy at any point of the disturbance in a crystalline medium. We have

$$T = \tfrac{1}{2}\int(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)\, d\tau, \quad \text{and} \quad W = \tfrac{1}{2}\int(a^2f^2 + b^2g^2 + c^2h^2)\, d\tau.$$

As (f, g, h) is the curl of (ξ, η, ζ) we may write W in the form

$$W = \tfrac{1}{2}\int\left[a^2 f\left(\frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\right) + b^2 g\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right) + c^2 h\left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial y}\right)\right] d\tau,$$

whence, integrating by parts, we get

$$2W = \int\left[\xi\left(c^2\frac{\partial h}{\partial y} - b^2\frac{\partial g}{\partial z}\right) + \eta\left(a^2\frac{\partial f}{\partial z} - c^2\frac{\partial h}{\partial x}\right) + \zeta\left(b^2\frac{\partial g}{\partial x} - a^2\frac{\partial f}{\partial y}\right)\right] d\tau$$
$$- \int[l_1(b^2g\zeta - c^2h\eta) + m_1(c^2h\xi - a^2f\zeta) + n_1(a^2f\eta - b^2g\xi)]\, dS,$$

where (l_1, m_1, n_1) are the direction cosines of the outward normal to the element dS of the surface bounding the region under consideration.

From the dynamical equations we have $\ddot{\xi} = b^2\dfrac{\partial g}{\partial z} - c^2\dfrac{\partial h}{\partial y}$, so that if $p/2\pi$ be the frequency, we have $p^2\xi = c^2\dfrac{\partial h}{\partial y} - b^2\dfrac{\partial g}{\partial z}$, with similar equations for η and ζ. Hence

$$\xi\left(c^2\frac{\partial h}{\partial y} - b^2\frac{\partial g}{\partial z}\right) + \eta\left(a^2\frac{\partial f}{\partial z} - c^2\frac{\partial h}{\partial x}\right) + \zeta\left(b^2\frac{\partial g}{\partial x} - a^2\frac{\partial f}{\partial y}\right)$$
$$= p^2(\xi^2 + \eta^2 + \zeta^2) = -(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2).$$

If then E be the whole energy, so that $E = T + W$, we have

$$E = -\tfrac{1}{2}\int[l_1(b^2 g\zeta - c^2 h\eta) + m_1(c^2 h\xi - a^2 f\zeta) + n_1(a^2 f\eta - b^2 g\xi)]\,dS.$$

From the formula on pp. 146 and 132 we have $h_1 vr = v^2 l - \kappa\lambda$, and $v^2 - a^2 = -\kappa l/\lambda$, so that

$$v^2\lambda = a^2\lambda - \kappa l, \quad v^2\mu = b^2\mu - \kappa m, \quad v^2\nu = c^2\nu - \kappa n.$$

Multiplying the second of these by N, the third by M, and subtracting, we get

$$v^2(\mu N - \nu M) = b^2\mu N - c^2\nu M + \kappa(Mn - Nm).$$

But the lines (l, m, n), (λ, μ, ν), and (L, M, N) are mutually at right angles, so that we have $l = \mu N - \nu M$, $\lambda = Mn - Nm$, and other similar equations. Hence

$$v^2 l = b^2\mu N - c^2\nu M + \kappa\lambda, \text{ and } h_1 vr = b^2\mu N - c^2\nu M.$$

If then, as before, A is the amplitude of (ξ, η, ζ), and B of the curl (f, g, h), we have $b^2 g\zeta - c^2 h\eta = AB(b^2\mu N - c^2\nu M) = L_1 vrAB$. We thus get

$$E = -\tfrac{1}{2}\int ABvr(L_1 l_1 + M_1 m_1 + N_1 n_1)\,dS = -\tfrac{1}{2}\int ABvr\cos\epsilon\,.\,dS$$

where ϵ is the angle between the *ray* (L_1, M_1, N_1) and the normal to the bounding surface (l_1, m_1, n_1). The rate at which energy is flowing out of this surface is $-\dfrac{dE}{dt} = ipE$, so that this rate is proportional to $\int ABvr\cos\epsilon\,.\,dS$.

Since the component of the flux along the normal to the surface is proportional to the cosine of the angle that this normal makes with the ray, we see that the energy may be regarded as flowing along the ray. Thus, with crystalline as well as with isotropic media, the two definitions of a ray suggested in Chapter II. are really identical.

CHAPTER VII.

REFLECTION AND REFRACTION WITH TRANSPARENT CRYSTALS.

In the last chapter we have dealt with the problem of crystalline reflection and refraction in so far as we have obtained geometrical relations between the wave normal, the refracted ray, and the direction of displacement. The results thus obtained will be required now, when we seek a fuller solution of the problem, and attempt to find the intensity and azimuth of the displacement for all the rays involved. This is solved in the same way as the corresponding problem for isotropic media, by making use of the boundary conditions suggested by the fundamental Principle of Action. We have seen that these boundary conditions are satisfied when (ξ, η, ζ) and the tangential components of the vector (a^2f, b^2g, c^2h) are continuous at an interface, and it will appear on trial that these conditions lead to only four independent equations.

If A be the amplitude of the displacement (ξ, η, ζ), so that for progressive waves A^2 is proportional to the intensity of the light, we have, in the notation already adopted,

$$\xi = ALe^{i\kappa'(lx+my+nz-vt)} = ALe^{i\omega}, \quad \eta = AMe^{i\omega}, \quad \text{and} \quad \zeta = ANe^{i\omega}.$$

The vector (f, g, h) is, as we have seen, intimately associated with the displacement, so that we have

$$(f, g, h) = (\lambda, \mu, \nu)\, Be^{i(\omega-\pi/2)} = (\lambda, \mu, \nu)\,(p/v)\, Ae^{i(\omega-\pi/2)}.$$

Let accented letters, such as A' and λ', refer to the reflected wave, and subscripts, such as A_1 and λ_1 or A_2 and λ_2, refer to the refracted waves. The boundary conditions require ξ, η, and ζ to be continuous at an interface for all values of t and at all points of the interface. This necessitates the relations $\omega = \omega' = \omega_1 = \omega_2$, so that

there is no change of phase in the ideal case of an abrupt transition. These relations require

$$\sin\phi/v = \sin\phi'/v = \sin\phi_1/v_1 = \sin\phi_2/v_2 = q \text{ say,}$$

just as in the case of isotropic media, ϕ being the angle that the wave normal makes with the normal to the interface. The equations thus obtained give the laws of reflection and refraction as far as the direction of the waves is concerned. The calculation of v_1 and v_2 is carried out in the last chapter, as well as the discussion of the relations between the wave normals and the corresponding *rays*. The intensities in the different waves are obtained by means of the boundary conditions. These boundary conditions involve the vector (a^2f, b^2g, c^2h), and we begin by obtaining the components of this vector in certain directions convenient for our purpose.

Resolving along the wave normal, we get

$$\begin{aligned}(p/v)\,Ae^{i(\omega-\pi/2)}\,(a^2l\lambda + b^2m\mu + c^2n\nu) &= (p/v)\,Ae^{i(\omega-\pi/2)}.\,\kappa\\ &= pvA\tan\chi.e^{i(\omega-\pi/2)}\end{aligned}$$

(where χ is the angle between the ray and the wave normal)

$$= (p/q)\,A\sin\phi\tan\chi.e^{i(\omega-\pi/2)}.$$

Resolving along the direction of the curl, we get

$$\begin{aligned}(p/v)\,(a^2\lambda^2 + b^2\mu^2 + c^2\nu^2)\,Ae^{i(\omega-\pi/2)} &= pvAe^{i(\omega-\pi/2)}\\ &= (p/q)\,A\sin\phi e^{i(\omega-\pi/2)},\end{aligned}$$

and finally resolving along the direction of displacement, we get

$$(p/v)\,(a^2\lambda L + b^2\mu M + c^2\nu N)\,Ae^{i(\omega-\pi/2)} = 0,$$

as was proved on p. 133. From these results it appears that the vector (a^2f, b^2g, c^2h) has a resultant $(p/q)\,A\sin\phi\sec\chi e^{i(\omega-\pi/2)}$ at right angles to the ray, in the plane of the ray, the curl, and the wave normal. Hence if OB be any line in the interface, ON the wave normal, and OP the direction of the curl, the boundary condition requiring the continuity of the component of (a^2f, b^2g, c^2h) in the direction OB leads to the equation

$$\begin{aligned}&A\sin\phi\cos BP + A'\sin\phi\cos BP'\\ &\quad= A_1\sin\phi_1\cos BP_1 + A_2\sin\phi_2\cos BP_2\\ &\qquad\quad + A_1\sin\phi_1\tan\chi_1\cos BN_1 + A_2\sin\phi_2\tan\chi_2\cos BN_2{}^*.\end{aligned}$$

* χ is here taken positively when the wave normal lies between the ray and the positive direction of the curl.

In Fig. 61, $x=0$ represents the interface, $z=0$ the plane of incidence, TOP the wave front, ON the wave normal, and OT the

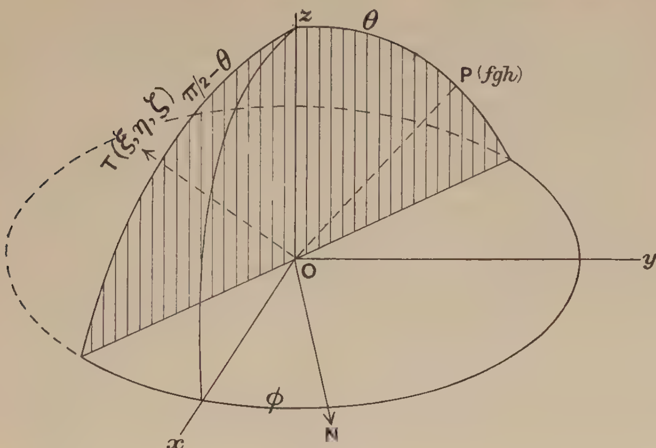

Fig. 61.

direction of the displacement, OP that of the curl—the arrow-heads indicating the sense which is to be regarded as positive. Let $POz=\theta$, and $xON=\phi$ (the angle of incidence). The plane through the wave normal and the direction of the displacement is the plane of polarisation NOT, so that θ is the angle between the planes of polarisation and incidence. We have

$$L=\cos\theta\sin\phi,\quad M=-\cos\theta\cos\phi,\quad N=\sin\theta,$$

and if B be a point on the axis of y, $\cos BP=\sin\theta\cos\phi$, and $\cos BN=\sin\phi$. In the refracted waves we must replace ϕ by ϕ_1 (or ϕ_2), and θ by θ_1 (or θ_2). In the reflected wave ϕ must be replaced by $\pi-\phi$ from the law of reflection, and θ by $-\theta'$. That $-\theta'$ is to take the place of θ in the reflected wave is seen on considering the case of direct incidence. In this case N and N' are on opposite sides of the wave front, so that θ and θ' are measured in opposite senses, so that, with our convention as to signs, if OT be above the plane $z=0$, OT' is below it.

We are now in a position to set out the boundary conditions in a convenient form. The continuity of ξ, η, ζ gives us

$$LA+L'A'=L_1A_1=L_2A_2$$

and two similar equations, while a fourth independent condition

is obtained by expressing the continuity of the component of $(a^2 f,\ b^2 g,\ c^2 h)$ along OB. We thus get the four equations

$$\begin{aligned}
(A\cos\theta + A'\cos\theta')\sin\phi &= A_1\cos\theta_1\sin\phi_1 + A_2\cos\theta_2\sin\phi_2,\\
(A\cos\theta - A'\cos\theta')\cos\phi &= A_1\cos\theta_1\cos\phi_1 + A_2\cos\theta_2\cos\phi_2,\\
A\sin\theta - A'\sin\theta' &= A_1\sin\theta_1 + A_2\sin\theta_2,\\
(A\sin\theta + A'\sin\theta')\sin 2\phi &= A_1(\sin\theta_1\sin 2\phi_1 + 2\tan\chi_1\sin^2\phi_1)\\
&\quad + A_2(\sin\theta_2\sin 2\phi_2 + 2\tan\chi_2\sin^2\phi_2).
\end{aligned}$$

When the incident wave is completely specified A, θ, and ϕ are given, and thence θ_1, θ_2, ϕ_1, ϕ_2, χ_1, χ_2 are known from the properties of the wave surface dealt with in the last chapter. There are thus just sufficient equations to determine θ' and the unknown amplitudes A', A_1, and A_2.

From these four equations we see that it will be possible by a proper choice of the direction of displacement (θ) in the incident wave to make one or other of the refracted waves (A_1 or A_2) disappear. Suppose, e.g., that we wish to make the wave A_2 disappear, then writing $\sin^2\phi_1\tan\chi_1/\sin\theta_1 = f_1$ for brevity, we have

$$\begin{aligned}
A\cos\theta + A'\cos\theta' &= A_1\cos\theta_1\sin\phi_1/\sin\phi,\\
A\cos\theta - A'\cos\theta' &= A_1\cos\theta_1\cos\phi_1/\cos\phi,\\
A\sin\theta - A'\sin\theta' &= A_1\sin\theta_1,\\
A\sin\theta + A'\sin\theta' &= A_1\sin\theta_1(\sin\phi_1\cos\phi_1 + f_1)/\sin\phi\cos\phi.
\end{aligned}$$

These give

$$\begin{aligned}
2A\cos\theta &= A_1\cos\theta_1\sin(\phi+\phi_1)/\sin\phi\cos\phi,\\
2A'\cos\theta' &= -A_1\cos\theta_1\sin(\phi-\phi_1)/\sin\phi\cos\phi,\\
2A\sin\theta &= A_1\sin\theta_1[\sin(\phi+\phi_1)\cos(\phi-\phi_1) + f_1]/\sin\phi\cos\phi,\\
2A'\sin\theta' &= -A_1\sin\theta_1[\cos(\phi+\phi_1)\sin(\phi-\phi_1) - f_1]/\sin\phi\cos\phi,
\end{aligned}$$

whence $$\tan\theta = \tan\theta_1[\cos(\phi-\phi_1) + f_1/\sin(\phi+\phi_1)]$$

and $$\tan\theta' = \tan\theta_1[\cos(\phi+\phi_1) - f_1/\sin(\phi-\phi_1)].$$

These equations correspond to those obtained on pp. 44 and 46 for isotropic media, where $\chi_1 = 0$, and therefore $f_1 = 0$. The equation $\tan\theta = \tan\theta_1[\cos(\phi-\phi_1) + f_1/\sin(\phi+\phi_1)]$ determines a definite direction for the incident displacement in order that the second refracted wave (A_2) may disappear, and by interchanging the subscripts 1 and 2 we should get a similar condition for the disappearance of

the other refracted wave. The two directions of the displacement in the incident wave, corresponding to a single refracted wave, are termed *uniradial directions.* The relative magnitudes of the different displacements for a uniradial system are determined by the equations obtained above.

We have seen that the displacement (ξ, η, ζ) is a vector whose square is proportional to the energy per unit volume in any medium, and that in passing from one medium to another the components of this vector are continuous. It follows from this that the refracted displacements (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) are equivalent, in the mechanical sense, to the displacements in the first medium, i.e. to the displacements in the incident and reflected waves. As a special case it is to be noted that when the incident displacement coincides with a uniradial direction then the refracted displacement is the resultant, in the mechanical sense, of the other two, so that the three displacements are in the same plane and are connected by the parallelogram law.

We proceed to investigate the position of this plane that contains the three displacements, and to prove that it coincides

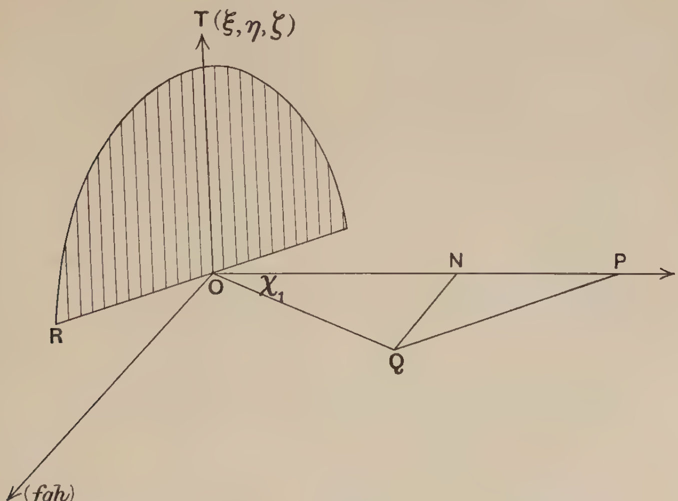

Fig. 62.

with what is known as the polar plane of the refracted ray. The polar plane of a ray is defined as follows. Let OQ (Fig. 62)

be the ray, ONP the corresponding wave normal, P a point on the index surface, so that $ON . OP = 1$. Let OT be the direction of the displacement and draw OR through O parallel to PQ.

Then the plane ROT is called the polar plane of the ray. In the special case in which the ray coincides with the wave normal, as happens with an isotropic medium, or with the ordinary ray in a uniaxal crystal, the polar plane becomes the plane through the normal and the displacement, i.e. the plane of polarisation of the ray.

The angle between the wave normal and the polar plane of the refracted ray is OPQ, and $\tan OPQ = QN/NP$. Also we have $ON = v_1 = 1/OP$, and therefore $ON/OP = v_1^2 = \sin^2\phi_1/\sin^2\phi$. Hence

$$\frac{\sin^2\phi_1}{\sin^2\phi - \sin^2\phi_1} = \frac{ON}{OP - ON} = \frac{ON}{NP} = \frac{ON}{QN} \cdot \frac{QN}{NP} = \cot\chi_1 . \tan OPQ,$$

where χ_1 is the angle between the ray and the wave normal. Thus we get

$$\begin{aligned} \tan OPQ &= \tan\chi_1 . \sin^2\phi_1/(\sin^2\phi - \sin^2\phi_1) \\ &= f_1 \sin\theta_1/\sin(\phi + \phi_1)\sin(\phi - \phi_1). \end{aligned}$$

Now let $z = px + qy$ be the equation of the plane of the three displacements when the incident displacement is in a uniradial direction. This equation is satisfied by the coordinates of any point on the vector representing the incident displacement, and also any point on that representing the reflected displacement. The coordinates of the former point are proportional to L, M, N, i.e. to $\cos\theta\sin\phi$, $-\cos\theta\cos\phi$, and $\sin\theta$, while those of the latter are proportional to L', M', N', i.e. to $\cos\theta'\sin\phi$, $\cos\theta'\cos\phi$, and $-\sin\theta'$. Substituting in the equation of the plane, and using the values of $\tan\theta$ and $\tan\theta'$ already obtained on p. 160 we get

$$\begin{aligned} p &= \tan\theta_1[\sin\phi_1 + f_1\cos\phi_1/\sin(\phi + \phi_1)\sin(\phi - \phi_1)] \\ &= \tan\theta_1 \sec\psi_1 \sin(\phi_1 + \psi_1), \end{aligned}$$

and

$$\begin{aligned} q &= -\tan\theta_1[\cos\phi_1 - f_1\sin\phi_1/\sin(\phi + \phi_1)\sin(\phi - \phi_1)] \\ &= -\tan\theta_1 \sec\psi_1 \cos(\phi_1 + \psi_1), \end{aligned}$$

where $$\tan\psi_1 = f_1/\sin(\phi + \phi_1)\sin(\phi - \phi_1).$$

Hence the equation of the plane becomes

$$z = px + qy = \tan\theta_1 \sec\psi_1 [x \sin(\phi_1 + \psi_1) - y \cos(\phi_1 + \psi_1)],$$

so that the plane cuts the plane of incidence ($z = 0$) in a line making an angle $\phi_1 + \psi_1$ with the axis of x.

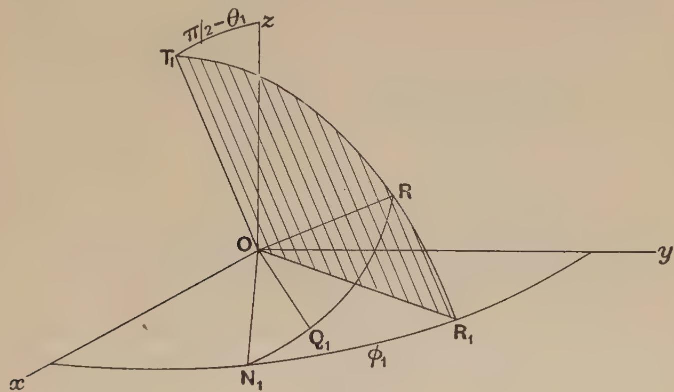

Fig. 63.

Next consider the polar plane R_1OT_1 (Fig. 63) of the refracted ray, R_1 being the point on the unit sphere where the line OR of Fig. 62 meets the sphere, and so for the other points. Let this polar plane meet the plane $z = 0$ in the line OK_1. T_1 is the pole of the plane $N_1Q_1R_1$, and z is the pole of N_1K_1, hence $\pi/2 - \theta_1$ = angle T_1Oz = angle between the planes of which T_1 and z are the poles = $R_1N_1K_1$. Thus from the triangle $R_1N_1K_1$ we get $\sin\theta_1 = \tan R_1N_1 \cot N_1K_1$.

But $\tan R_1N_1 = \tan R_1ON_1 = \tan RON = \tan OPQ$

$$= f_1 \sin\theta_1 / \sin(\phi + \phi_1) \sin(\phi - \phi_1) = \sin\theta_1 \tan\psi_1,$$

so that we have $\sin\theta_1 \tan\psi_1 = \tan R_1N_1 = \sin\theta_1 \tan N_1K_1$. Hence $N_1K_1 = \psi_1$, and OK_1 makes an angle $\phi_1 + \psi_1$ with the axis of x.

We have thus proved that the polar plane of the refracted ray and the plane of the three displacements have the line OK_1 in common, and as they also have the line OT_1 in common, the two planes must coincide. As the displacement is in the wave front, it follows from this theorem that the displacements in the incident and reflected waves are along the lines of intersection of these waves with the polar plane of the refracted ray.

These theorems with reference to a uniradial system enable us to solve the most general problem even when the incident displacement is not in a uniradial direction. For suppose we have an incident displacement of amplitude A in any azimuth θ. We

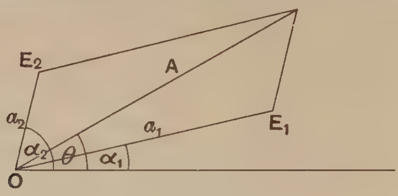

Fig. 64.

can resolve this displacement into two displacements whose amplitudes are a_1 and a_2 along the uniradial directions OE_1 and OE_2, the azimuths of these displacements being α_1 and α_2 (say). We may then apply the results already reached to obtain the displacements in the reflected wave. Their directions are the intersections of the reflected wave with the polar planes of the refracted rays corresponding to each uniradial direction. Denoting their lengths by a_1' and a_2' and their azimuths by α_1' and α_2', we can compound them by the parallelogram law into a single displacement A' in the azimuth θ'. Of this general construction there is one case of special interest. This occurs when the reflected wave passes through the line of intersection of the two polar planes. In these circumstances there is only one possible direction of the reflected displacement whatever be the azimuth of the incident displacement, this direction being the line common to the two polar planes. In this case, therefore, the reflected light is plane polarised, and, as the reflected ray is perpendicular to its wave front, the ray is at right angles to the intersection of the two polar planes—a relation corresponding to Brewster's law that the incident and refracted rays are at right angles at the polarising angle for isotropic media. When the two reflected displacements coincide we have $\theta' = \beta'$, where

$$\tan \beta' = \tan \theta_1 [\cos(\phi + \phi_1) - f_1 \operatorname{cosec}(\phi - \phi_1)]$$
$$= \tan \theta_2 [\cos(\phi + \phi_2) - f_2 \operatorname{cosec}(\phi - \phi_2)].$$

This is an equation to determine the polarising angle and also the value of θ' at this angle. It thus appears that when a ray is

polarised by reflection from a crystal the plane of polarisation deviates by an angle β' from the plane of incidence. The angle β' is therefore called the *deviation*.

From the equations on p. 160 we see that

$$\frac{a_1}{a_1'} = -\frac{\cos\alpha_1' \sin(\phi+\phi_1)}{\cos\alpha_1 \sin(\phi-\phi_1)}, \quad \text{and} \quad \frac{a_2}{a_2'} = -\frac{\cos\alpha_2' \sin(\phi+\phi_2)}{\cos\alpha_2 \sin(\phi-\phi_2)}.$$

Also, from the parallelogram law, we have

$$\frac{a_1}{a_2} = \frac{\sin(\alpha_2-\theta)}{\sin(\theta-\alpha_1)}, \quad \frac{a_1'}{a_2'} = \frac{\sin(\alpha_2'-\theta')}{\sin(\theta'-\alpha_1')}.$$

From these equations we derive

$$\frac{a_1 a_2'}{a_2 a_1'} = \frac{\sin(\phi+\phi_1)\sin(\phi-\phi_2)\cos\alpha_1'\cos\alpha_2}{\sin(\phi+\phi_2)\sin(\phi-\phi_1)\cos\alpha_1\cos\alpha_2'}$$

$$= \frac{\sin(\alpha_2-\theta)\sin(\theta'-\alpha_1')}{\sin(\alpha_2'-\theta')\sin(\theta-\alpha_1)}.$$

But $$\frac{\cos\alpha_1'\sin(\theta-\alpha_1)}{\cos\alpha_1\sin(\theta'-\alpha_1')} = \frac{\cos\theta}{\cos\theta'}\,\frac{\tan\theta-\tan\alpha_1}{\tan\theta'-\tan\alpha_1'};$$

and $$\frac{\cos\alpha_2'\sin(\alpha_2-\theta)}{\cos\alpha_2\sin(\alpha_2'-\theta')} = \frac{\cos\theta}{\cos\theta'}\,\frac{\tan\theta-\tan\alpha_2}{\tan\theta'-\tan\alpha_2'}.$$

Whence

$$C = \frac{\sin(\phi+\phi_1)\sin(\phi-\phi_2)}{\sin(\phi+\phi_2)\sin(\phi-\phi_1)} = \frac{\tan\theta-\tan\alpha_2}{\tan\theta'-\tan\alpha_2'} \times \frac{\tan\theta'-\tan\alpha_1'}{\tan\theta-\tan\alpha_1}$$

$$= \frac{t-t_2}{t'-t_2'} \times \frac{t'-t_1'}{t-t_1},$$

where, for brevity, t is written for $\tan\theta$, and so for the others.

Thus we have

$$(1-C)\,tt' + (Ct_2'-t_1')\,t + (Ct_1-t_2)\,t' + t_2t_1' - Ct_1t_2' = 0,$$

which is of the form

$$ptt' + qt + rt' + s = 0,$$

where p, q, r, and s are constants for a given angle of incidence. Thus there is a homographic relation between t and t', as might have been anticipated from the fact that there is one value of θ' for each value of θ, and *vice versa*. This homographic relation shows that, for a given angle of incidence, the intersection of the planes of polarisation of the incident and reflected rays describes a cone of the second degree.

Just as with an ordinary homographic range the relation between the conjugates can be simplified by a proper choice of origins, so the homographic relation found above can be put in a simpler form. It can in fact be replaced by

$$\tan(\theta' - \beta') = \gamma \,.\, \tan(\theta - \beta),$$

where

$$(\gamma \tan\beta' - \tan\beta)/p = (\gamma + \tan\beta\tan\beta')/q$$
$$= -(1 + \gamma\tan\beta\tan\beta')/r = (\tan\beta' - \gamma\tan\beta)/s.$$

From these we get

$$(p - s)/(r + q) = \tan(\beta + \beta'),$$

and

$$(p + s)/(r - q) = \tan(\beta - \beta').$$

These give β and β' in terms of p, q, r, and s, and γ is then obtained from the equation

$$\gamma = (p\tan\beta' + s\tan\beta)/(p\tan\beta + s\tan\beta').$$

It appears from the equation

$$\tan(\theta' - \beta') = \gamma\tan(\theta - \beta)$$

just obtained, or from the geometrical discussion on p. 164, that the azimuth (θ') of the reflected displacement will, in general, depend on the azimuth (θ) of the incident displacement. If, however, we have $\gamma = 0$, then $\theta' = \beta'$ whatever be the incident azimuth, and in this case the reflected ray will be plane polarised in the azimuth β'. On putting $\gamma = 0$ in the relations given above connecting p, q, r, and s, and eliminating β and β', we get $ps = qr$,

i.e.
$$(1 - C)(t_2 t_1' - C t_1 t_2') = (C t_2' - t_1')(C t_1 - t_2),$$

or
$$C(t_1 - t_2)(t_1' - t_2') = 0.$$

As C cannot vanish, and t_1 is not equal to t_2, we see that $t_1' = t_2'$ or $\alpha_1' = \alpha_2'$. Hence when $y = 0$ we have $\theta' = \beta' = \alpha_1' = \alpha_2'$, which is the relation obtained on p. 164, for determining the polarising angle and the deviation.

The azimuths θ and θ' are measured from the plane of incidence; but it appears from the equation

$$\tan(\theta' - \beta') = \gamma\tan(\theta - \beta)$$

that the formulae will be somewhat simplified by measuring the incident azimuth from the line $\theta = \beta$, and the reflected azimuth from the line $\theta' = \beta'$. The relation between the azimuths then

takes the form $\tan\theta' = \gamma . \tan\theta$, so that $\theta' = 0$ when $\theta = 0$, and $\theta' = 90^\circ$ when $\theta = 90^\circ$. Everything then corresponds closely with the case of an isotropic medium, the azimuths $\theta = 0$ and 90° corresponding with directions parallel and perpendicular to the plane of incidence in the simpler case. In particular the investigation of p. 44 holds in the case of crystals, so that if we vary the incident azimuth, but keep the amplitude constant, the end of the line representing the reflected displacement traces out an ellipse, and the area described by the reflected displacement is proportional to that described by the incident displacement. These relations between the azimuths of the displacement in the incident and reflected light were compared with experimental determinations of them by Neumann. His experimental methods precluded very great accuracy, but the number of observations at each incidence was very large. The differences between the results of observation and the calculations from theory rarely exceeded 20′ and generally were less than 5′, such differences being well within the limits of errors in the experiments.

The formulae are, of course, somewhat simplified when the crystal is uniaxal, and as this case is one of special interest to the experimenter it will be worth considering in some detail.

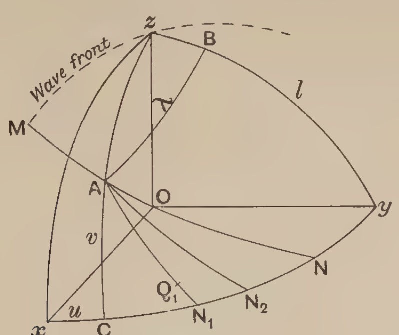

Fig. 65.

In Fig. 65 let $x = 0$ be the interface; $z = 0$ the plane of incidence; ON the normal to the incident wave, coinciding with the incident ray; OQ_1 the extraordinary refracted ray; OA the optic axis. The position of this axis is determined with reference to the coordinate planes by the coordinates (l, λ) or (u, v), where

AB being perpendicular to yz, $yB = l$ and $AB = \lambda$, and AC being perpendicular to xy, $Cx = u$ and $AC = v$.

We shall use the suffix 1 when dealing with the extraordinary refracted wave, and the suffix 2 when dealing with the ordinary refracted wave. If ω be the angle that the wave normal makes with the optic axis, we have from the formulae of last chapter,

$$v_1^2 = a^2 \sin^2 \omega_1 + c^2 \cos^2 \omega_1 = c^2 + (a^2 - c^2) \sin^2 \omega_1,$$

$$v_2 = c,$$

and $$1/\sin \phi = v_1/\sin \phi_1 = v_2/\sin \phi_2 = c/\sin \phi_2.$$

These equations determine ϕ_1 and ϕ_2 in terms of ϕ and ω_1. For the ordinary ray the wave normal and the ray coincide, so that $\chi_2 = 0$, and therefore $f_2 = 0$. Hence the equations on p. 160, which give the uniradial directions, reduce to

$$\tan \theta = \tan \theta_2 \cos (\phi - \phi_2), \text{ and } \tan \theta' = \tan \theta_2 \cos (\phi + \phi_2).$$

For the extraordinary ray we have

$$\begin{aligned}\tan \chi_1 &= (a^2 - c^2) \sin \omega_1 \cos \omega_1 / v_1^2 \\ &= (a^2 - c^2) \sin \omega_1 \cos \omega_1 \sin^2 \phi / \sin^2 \phi_1,\end{aligned}$$

and

$$f_1 = \sin^2 \phi_1 \tan \chi_1 / \sin \theta_1 = (a^2 - c^2) \sin \omega_1 \cos \omega_1 \sin^2 \phi / \sin \theta_1.$$

Hence in this case

$$\begin{aligned}\tan \theta &= \tan \theta_1 \cos (\phi - \phi_1) \\ &\qquad + (a^2 - c^2) \sin \omega_1 \cos \omega_1 \sin^2 \phi / \cos \theta_1 \sin (\phi + \phi_1),\end{aligned}$$

and

$$\begin{aligned}\tan \theta' &= \tan \theta_1 \cos (\phi + \phi_1) \\ &\qquad + (a^2 - c^2) \sin \omega_1 \cos \omega_1 \sin^2 \phi / \cos \theta_1 \sin (\phi - \phi_1).\end{aligned}$$

To put these equations, giving the uniradial directions, in a form suitable for comparison with experiment we must obtain the relations between θ_1, θ_2 and ϕ_1, ϕ_2. The latter (ϕ_1 and ϕ_2) can be determined by observing the deviation produced by a prism. It is obvious from symmetry that the two possible directions of the displacement in the wave front are in the meridional plane of the crystal and at right angles thereto. Hence if AM (Fig. 65) be perpendicular to the wave front, the directions of displacement are

along OM and at right angles to this. From the triangle AzM we get

$$\cos AzM = \tan zM \cot zA,$$

i.e.

$$\sin(\phi_2 \sim u) = + \cot\theta_2 \tan v,$$

and

$$\sin(\phi_1 \sim u) = - \tan\theta_1 \tan v.$$

Also from the triangle ACN_1 we have

$$\cos AN_1 = \cos AC \cos CN,$$

i.e.

$$\cos\omega_1 = \cos v \cos(\phi_1 \sim u),$$

whilst from AzM we have

$$\cos AM \cos LM = \cos Az,$$

i.e.

$$\sin\omega_1 \cos\theta_1 = \sin v.$$

These equations give ω_1, θ_1 and θ_2 in terms of ϕ_1 and ϕ_2 when u and v are known, i.e. when the position of the optic axis is known with reference to the coordinate planes.

The azimuths of the displacements in a uniradial system were observed by Neumann for a uniaxal crystal, Iceland spar. The differences between theory and experiment varied from $-8'$ to $+4'$, and these differences were scarcely greater than the probable errors of the experiments. The range of incidence, however, was small, the values of ϕ employed being 40°, 45°, 50°, and 55°. The more modern and more accurate experiments of Glazebrook confirm Neumann's results by showing a very good agreement between theory and observation for certain incidences. For other incidences, however, there is a small discrepancy, so that the theory can be relied upon as colligating the experimental results only as a close approximation. Instead of attempting a comparison with theory by direct measurements of θ, Glazebrook proceeded as follows. He allowed plane polarised light to fall on a prism of spar, and varied the angle of incidence until only one ray emerged. He then measured ϕ and the deviation produced by the prism, and from these observations ϕ_1 or ϕ_2 could easily be determined. He thus had sufficient data to calculate θ by means of the formulae obtained above. The incident light was then made to pass through a sugar cell before falling on the prism, and the incidence was again varied until the same ray emerged as before, θ being once more calculated from theory, the change from the former

value gave the rotation produced by the cell. This rotation could be determined accurately by experiment, and proved to be 4° 6′ 15″ when only the ordinary ray was transmitted, and 4° 4′ 20″ when only the extraordinary ray traversed the crystal. The results obtained are given in the following tables, in which the first line gives the incidence before and the second that after the interposition of the sugar cell. The third line gives the difference between the rotation as calculated from theory and as found by experiment. Table I refers to the case where only the ordinary ray traversed the crystal, and Table II deals similarly with the extraordinary ray.

TABLE I.

ϕ	24° 42′	30° 36′	36° 5′ 40″	41° 41′ 30″	46° 50′ 40″	52° 16′ 40″	57° 54′ 20″
ϕ (with cell)	17° 45′	24° 25′	30° 18′ 50″	36° 4′ 10″	41° 30′ 50″	47° 1′ 40″	52° 24′ 30″
Difference between theory & exp.	+4′ 55″	−0′ 5″	+0′ 35″	+6′ 25″	−0′ 25″	−4′ 35″	−0′ 55″
ϕ	59° 45′	64° 12′ 30″	66° 6′ 20″	70° 57′ 40″	73° 43′	78° 14′	80° 47′
ϕ (with cell)	54° 8′ 30″	58° 9′ 30″	59° 51′ 50″	63° 57′ 50″	66° 6′ 20″	69° 29′	70° 59′
Difference between theory & exp.	−0′ 5″	+3′ 35″	+4′ 5″	+9′ 5″	+11′ 25″	+18′ 35″	+15′ 55″

TABLE II.

ϕ	24° 15′	29° 40′	35° 7′ 30″	40° 9′ 20″	45° 17′	50° 18′	55° 43′
ϕ (with cell)	17° 45′	24° 4′	29° 50′ 30″	35° 4′ 30″	40° 17′	45° 15′	50° 27′ 30″
Difference between theory & exp.	+13′ 50″	+4′ 50″	+5′ 20″	+6′ 20″	+4′ 20″	+2′ 0″	−5′ 10″
ϕ	61° 29′ 40″	64° 46′ 50″	66° 42′	69° 3′ 30″	73° 4′ 40″	77° 24′ 20″	82° 0′ 55″
ϕ (with cell)	55° 37′	58° 17′	59° 57′	61° 37′ 30″	64° 57′ 30″	67°	69° 21′ 30″
Difference between theory & exp.	−6′ 20″	−6′ 0″	−12′ 0″	−8′ 10″	−17′ 50″	−23′ 2″	−40′ 10″

Returning to the theory we see that we can determine the polarising angle and the deviation (β') by making θ' the same for the ordinary and extraordinary rays. We thus get

$$\tan \beta' = \tan \theta_2 \cos (\phi + \phi_2) = \tan \theta_1 \cos (\phi + \phi_1)$$
$$+ (a^2 - c^2) \sin \omega_1 \cos \omega_1 \sin^2 \phi / \cos \theta_1 \sin (\phi - \phi_1).$$

To solve this equation we must proceed by successive approximations, a process greatly simplified by the fact that in all actual cases the difference between a and c is small. Thus ϕ_1 and ϕ_2 are nearly equal. We have

$$\sin \phi_2 = c \sin \phi, \text{ and } \sin \phi_1 = v_1 \sin \phi,$$

and therefore

$$\sin^2 \phi_1 - \sin^2 \phi_2 = (v_1^2 - c^2) \sin^2 \phi = (a^2 - c^2) \sin^2 \omega_1 \sin^2 \phi.$$

Hence $\phi_1 - \phi_2 = \sin (\phi_1 - \phi_2)$ approximately

$$= (a^2 - c^2) \sin^2 \omega_1 \sin^2 \phi / \sin (\phi_1 + \phi_2) = (a^2 - c^2) \sin^2 \omega_1 \sin^2 \phi / \sin 2\phi_2.$$

We have seen that in the case of isotropic media, for which $\phi_1 = \phi_2$, the polarising angle occurs when $\phi + \phi_1 = \pi/2$. Thus when ϕ_1 and ϕ_2 are nearly equal we have $\phi + \phi_1 = \phi + \phi_2 = \pi/2$ approximately. Hence to this order of approximation

$$\phi - \phi_1 = \phi + \phi_1 - 2\phi_1 = \pi/2 - 2\phi_1 = \pi/2 - 2\phi_2,$$

and

$$\cos (\phi + \phi_1) = \cos (\overline{\phi + \phi_2} + \overline{\phi_1 - \phi_2}) = \cos (\phi + \phi_2) - (\phi_1 - \phi_2)$$
$$= \cos (\phi + \phi_2) - (a^2 - c^2) \sin^2 \omega_1 \sin^2 \phi / \sin 2\phi_2.$$

Moreover we have seen that the two waves corresponding to any wave normal are polarised at right angles to one another, so that, as the normals N_1 and N_2 (Fig. 65) are nearly coincident, we have $\theta_2 = \pi/2 + \theta_1$ nearly. Making these various substitutions in the equation for the polarising angle, we get

$$\cos (\phi + \phi_2) = (a^2 - c^2) \sin^2 \phi \sin \omega_1 \cos \theta_2 [\cos \omega_1 / \cos 2\phi_2$$
$$+ \sin \omega_1 \cos \theta_2 / \sin 2\phi_2].$$

The ordinary ray ON_2 is polarised in the principal section of the crystal OAN_2, so that θ_2 is the angle AN_2C of Fig. 65. Hence from the triangle AN_2C of that figure we have

$$\cos \omega_2 = \cos v \cos (\phi_2 - u), \quad \sin \omega_2 \cos \theta_2 = \cos v \sin (\phi_2 - u).$$

Substituting in the above equation for $\cos (\phi + \phi_2)$, and making

use, in the terms containing the small factor $(a^2 - c^2)$, of the approximations $\omega_2 = \omega_1$ and $\tan\phi_2 = \cos\phi = c$, we obtain the following equations to determine the polarising angle $\phi = J$:

$$\sin\phi_2 = c\sin J,$$

and

$$\cos(J + \phi_2) = K\cos^2 v\,(\sin^2 u - \sin^2\phi_2) = K\cos^2 v\,(\sin^2 J - \sin^2 u),$$

where $K = (a^2 - c^2)(1 + c^2)/2c(1 - c^2)$. Solving these equations by successive approximations, we get as the first approximation $J = I = \cot^{-1} c$, which is the polarising angle for an isotropic medium, and as the second approximation

$$J = I - \kappa\,.\cos^2 v\,(\sin^2 I - \cos^2 u),$$

where $\kappa = (a^2 - c^2)/2c(1 - c^2)$. As the value of J is not altered by changing the signs of u and v, it follows that the polarising angle remains the same when the reflecting face of the crystal is turned through 180° in its own plane, and this in spite of the fact that this process changes one of the angles of refraction and alters the situation of the refracted rays with respect to the optic axes. This is in accordance with Brewster's observations and the more accurate measurements of Conroy. The following table gives the differences observed by Conroy in the values of the polarising angle for azimuths u and $u + 180°$ in the case of reflection from Iceland spar in air.

u	1°	11°	21°	31°	47° 20′	51°	61°	71°	91°
Difference in J	+14′	+5′	−1′	−9′	+8′	0	−15′	−2′	+11′
u	101°	117° 20′	131°	141°	151°	161°	167° 20′	171°	177° 20′
Difference in J	−7′	+3′	0	−7′	+6′	+13′	0	+5′	−13′

If we wish to use the coordinates l and λ instead of u and v, we have merely to transform the equation for J by means of the trigonometrical relations $\sin v = \cos\lambda\sin l$, and $\sin\lambda = \cos u\cos v$. These give

$$\begin{aligned}\cos^2 v\,(\sin^2 I - \cos^2 u) &= (1 - \cos^2\lambda\sin^2 l)\sin^2 I - \sin^2\lambda \\ &= \sin^2 I - \sin^2\lambda - \cos^2\lambda\sin^2 l\sin^2 I \\ &= \cos^2 l\,[\sin^2 I - \sin^2\lambda] - \sin^2 l\cos^2 I\sin^2\lambda.\end{aligned}$$

Hence $J = I - \kappa \cos^2 l\,(\sin^2 I - \sin^2 \lambda) + \kappa \sin^2 l \cos^2 I \sin^2 \lambda$. If then J_1 and J_2 denote the values of J when the plane of incidence is parallel and perpendicular respectively to the principal section of the crystal, we get $J = J_1 \cos^2 l + J_2 \sin^2 l$, where

$$J_1 = I - \kappa(\sin^2 I - \sin^2 \lambda), \text{ and } J_2 = I + \kappa \cos^2 I \sin^2 \lambda.$$

The following table compares the deductions from these formulae with the results of Seebeck's experiments on a crystal for which $I = 58° 55'$ and $\kappa = 0{\cdot}1158$ radians $= 6° 38'$.

λ......	0° 25′	27° 2′	45° 29′	64° 1′ 30″	89° 47′
J_1 (theory)	54° 3′	55° 25′	57° 25′	59° 25′	60° 41′
J_1 (exp.)	54° 12′	55° 36′	57° 22′	59° 19′	60° 33′
Difference	−9′	−11′	+3′	+6′	+8′
J_2 (theory)	58° 55′	59° 17′	59° 48′	60° 23′	60° 41′
J_2 (exp.)	58° 56′	59° 4′	59° 48′	60° 15′	60° 33′
Difference	−1′	+13′	0	+8′	+8′

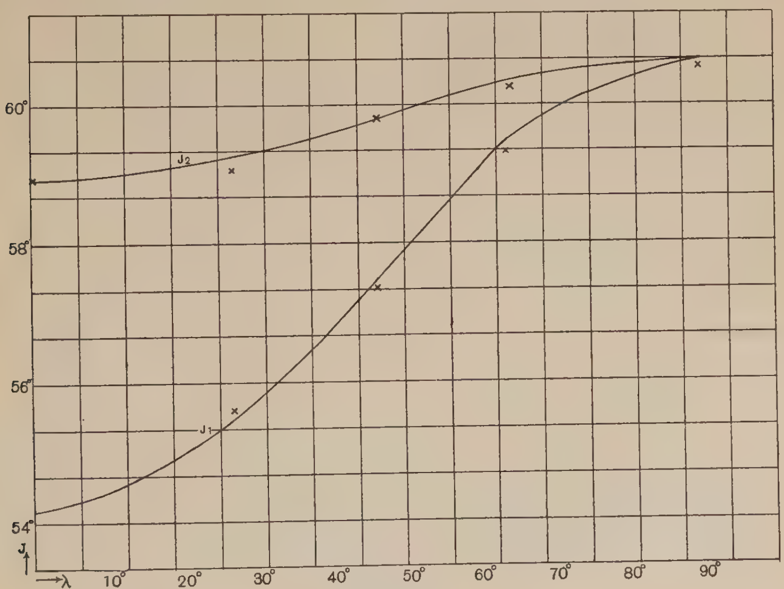

Fig. 66.

These results are shown graphically in Fig. 66, and from this figure or from the table it appears that the approximate formulae that have been employed in the calculations represented the facts very closely.

The mean of Brewster's observations on a crystal with a natural cleavage corresponding to $\lambda = 45° \; 23' \; 30''$ gave $J_1 = 57° \; 25'$ and $J_2 = 59° \; 38'$, while the values calculated from the above formulae are $J_1 = 57° \; 25'$ and $J_2 = 59° \; 48'$. The following table gives the values of J for different values of l, calculated from the formula $J = J_1 \cos^2 l + J_2 \sin^2 l$, and compared with Brewster's measurements.

l......	0	22° 30′	45°	67° 30′	90°
J (theory)	57° 25′	57° 46′	58° 36′	59° 27′	59° 48′
J (exp.)	57° 20′	57° 46′	58° 34′	59° 29′	59° 51′
Difference	+5′	0	+2′	−2′	−3′

Such small differences between theory and observation are within the limits of error of Brewster's experiments.

Although these approximate formulae have proved sufficiently accurate for our purposes, it may be well to obtain a rigorous solution of the problem under discussion in a case where such a solution can be easily reached. This happens when the optic axis lies in the plane of incidence. The crystal is then symmetrical on opposite sides of this plane, and the uniradial directions are obviously parallel and perpendicular thereto. We shall deal with the latter direction as representing the extraordinary ray.

In Fig. 67 let the plane of the paper represent the plane of incidence, OT the interface, and ON the wave normal. The incident and refracted rays OQ and OQ_1 are obtained from Huyghens' construction by drawing tangents to the wave surface, the section of which, in this case, is a circle of unit radius OQ, and an ellipse AQ_1C of which the axis OC makes an angle λ with OT. The argument employed on p. 50 for the case of an isotropic medium holds equally well here, so that QQ_1 is parallel to OT. The expression of this fact leads directly to a simple formula for the polarising angle.

We have $OQ = 1$, $OT = \operatorname{cosec} \phi$, and $QM = \cos \phi$, where QM is the perpendicular from Q on OT. Taking OA and OC as the

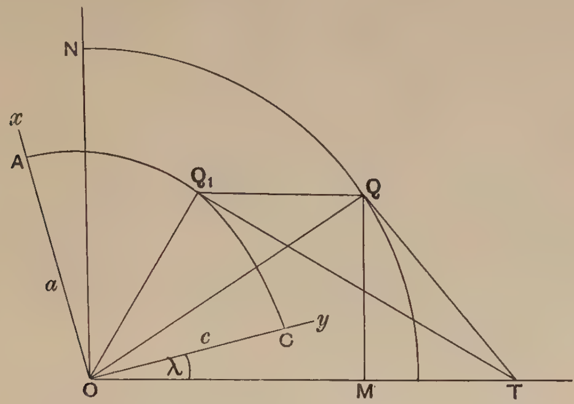

Fig. 67.

axes of x and y, and denoting the eccentric angle of the point Q_1 on the ellipse by ψ_1, the coordinates of Q_1 are $a \cos \psi_1$ and $c \sin \psi_1$, and the length of the perpendicular from this point on the line OT, whose equation is $x \cos \lambda + y \sin \lambda = 0$, is

$$a \cos \psi_1 \cos \lambda + c \sin \psi_1 \sin \lambda = QM = \cos \phi.$$

The equation to $Q_1 T$, the tangent to the ellipse, is

$$cx \cos \psi_1 + ay \sin \psi_1 = ac.$$

This line passes through T whose coordinates are $-\sin \lambda \operatorname{cosec} \phi$ and $\cos \lambda \sec \phi$, so that we have

$$a \sin \psi_1 \cos \lambda - c \cos \psi_1 \sin \lambda = ac \sin \phi.$$

Squaring this equation, and also $cx \cos \psi_1 + ay \sin \psi_1 = c \cos \phi$, and adding, we get

$$a^2 \cos^2 \lambda + c^2 \sin^2 \lambda = \cos^2 \phi + a^2 c^2 \sin^2 \phi = 1 - \sin^2 \phi\,(1 - a^2 c^2),$$

whence $\sin^2 I = \sin^2 \phi = (1 - a^2 \cos^2 \lambda - c^2 \sin^2 \lambda)/(1 - a^2 c^2)$, which determines the polarising angle.

Returning to the consideration of the approximate formulae, we find the deviation β' from the equation $\tan \beta' = \tan \theta_2 \cos (\phi + \phi_2)$, where $\phi = J$. But $\tan \theta_2 = + \tan v \,.\, \sin (\phi_2 \sim u)$, and

$$\cos (\phi + \phi_2) = - K \cos^2 v \sin (\phi_1 - u) \sin (\phi_2 + u).$$

Hence $\tan\beta' = -K\sin v\cos v\sin(\phi_2+u)$, and since

$$u+\phi_2 = u-\phi+\phi+\phi_2 = \pi/2-(\phi-u)$$

nearly, we get $\tan\beta' = -K\sin v\cos v\cos(\phi-u) = -K\sin v\cos\psi$, where $\psi = AN$ (Fig. 65) is the angle between the incident ray and the optic axis. Thus

$$\begin{aligned}\tan\beta' &= -K\cos\lambda\sin l\cos\psi\\ &= -K\cos\lambda\sin l(\sin\lambda\cos I+\cos\lambda\sin I\cos l).\end{aligned}$$

The following table gives the results of Seebeck's observations of the deviation for reflection from a face of natural cleavage, and also for a face parallel to the optic axis. Fig. 68 represents the results of theory and observation graphically, and shows that the argument is, on the whole, very satisfactory.

	l......	0	22° 30′	45°	67° 30′	90°
Face of natural cleavage	β' (theory)	0	2° 11′	3° 33′	3° 31′	2° 21′
	β' (exp.)	0	2° 9′	3° 38′	3° 34′	2° 30′
	Difference	0	+2′	−5′	−3′	−9′
Face parallel to axis	β' (theory)	0	2° 46′	3° 54′	2° 46′	0
	β' (exp.)	0	2° 46′	3° 57′	2° 43′	0
	Difference	0	0	−3′	+3′	0

The polarising angle dealt with above is the angle of incidence at which the reflected light is plane polarised, whatever be the azimuth of the polarisation of the incident light. If, however, this azimuth corresponds to a uniradial direction, there is only one refracted ray, and we may inquire what the angle of incidence must be in order that the reflected beam should be polarised in the plane of incidence or at right angles thereto. This angle will be called the *uniradial polarising angle* to distinguish it from the complete polarising angle dealt with hitherto. Its magnitude is determined by putting $\theta' = 0$ in the fundamental equation

$$\tan\theta' = \tan\theta_1[\cos(\phi+\phi_1) - f_1/\sin(\phi-\phi_1)]$$

of p. 160. We shall consider more particularly the case of a uniaxal crystal. If the refracted ray be the ordinary ray we have

$\chi_2 = 0$, and therefore $f_2 = 0$, so that the above equation gives $\cos(\phi + \phi_2) = 0$ or $\phi + \phi_2 = \pi/2$, which is Brewster's law. This result has been verified experimentally in the case of Iceland

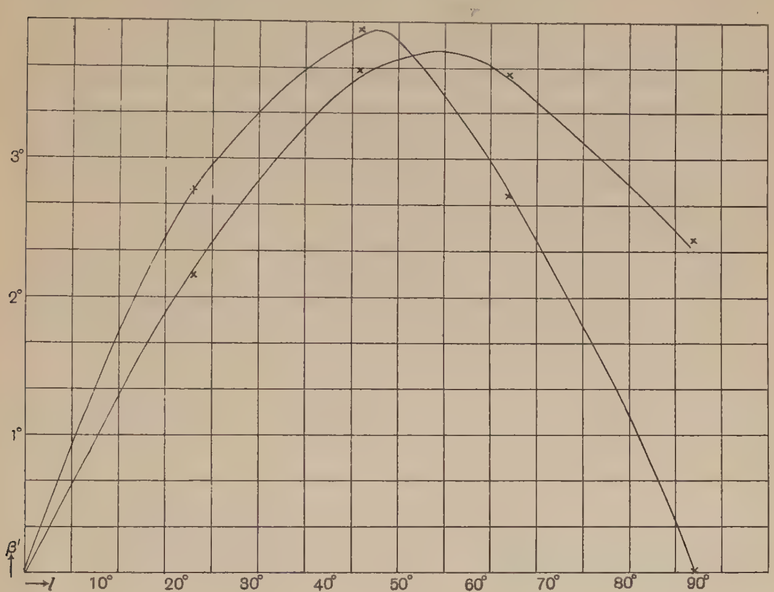

Fig. 68.

spar by Schmidt. He found the polarising angle for the ordinary ray to be independent of the orientation of the plane of incidence, and that for a series of observations in different azimuths the *greatest* departure from Brewster's law was only 16′. If, on the other hand, the refracted ray be the extraordinary one Brewster's law no longer holds. We then have

$$\cos(\phi + \phi_1)\sin(\phi - \phi_1) = f_1 = \sin^2\phi_1 \tan\chi_1/\sin\theta_1.$$

Also from Fig. 65 we have $\sin zM = \tan AM \,.\, \cot AzM$, so that $\sin\theta_1 \tan\omega_1 = \tan(u \sim \phi_1)$. Moreover

$$\begin{aligned}\tan\chi_1 &= (a^2 - c^2)\sin\omega_1\cos\omega_1/v_1^2 = (a^2 - c^2)\sin^2\omega_1/v_1^2\tan\omega_1 \\ &= (v_1^2 - c^2)/v_1^2\tan\omega_1 = (\sin^2\phi_1 - \sin^2\phi_2)/\sin^2\phi_1\tan\omega_1 \\ &= \sin(\phi_1 - \phi_2)\sin(\phi_1 + \phi_2)/\sin^2\phi_1\tan\omega_1.\end{aligned}$$

Hence

$$\cos(\phi + \phi_1)\sin(\phi - \phi_1) = \sin^2\phi_1 \tan\chi_1/\sin\theta_1$$
$$= \sin(\phi_1 - \phi_2)\sin(\phi_1 + \phi_2)/\sin\theta_1 \tan\omega_1$$
$$= \sin(\phi_1 - \phi_2)\sin(\phi_1 + \phi_2)\cot(u \sim \phi_1).$$

This equation, combined with $\sin\phi_2 = c\sin\phi$, enables us to find ϕ approximately by the method employed on p. 171. Since $\phi_1 - \phi_2$ is small, we see that $\cos(\phi + \phi_1)$ is small, so that $\phi + \phi_1$ is nearly $\pi/2$, and the polarising angle differs by a small quantity from that given by Brewster's law. We have, approximately,

$$\cos(\phi + \phi_1) = \cos(\phi + \phi_2) - (\phi_1 - \phi_2)\sin(\phi + \phi_2)$$
$$= \sin(\phi_1 - \phi_2)\sin(\phi_1 + \phi_2)\cot(u \sim \phi_1)/\sin(\phi - \phi_1)$$
$$= (\phi_1 - \phi_2)\sin 2\phi_2 \cot(u \sim \phi_2)/\sin(\phi - \phi_1).$$

So that

$$\cos(\phi + \phi_2) = (\phi_1 - \phi_2)[\sin(\phi + \phi_2) + \sin 2\phi_2 \cot(u \sim \phi_2)/\sin(\phi - \phi_2)],$$

also $$\phi_1 - \phi_2 = (a^2 - c^2)\sin^2\omega_1 \sin^2\psi/\sin 2\phi_2$$
$$= (a^2 - c^2)\sin^2\phi[1 - \cos^2 v\cos^2(u - \phi_2)]/\sin 2\phi_2.$$

Hence the polarising angle for the extraordinary ray is given by the formula

$$\phi = I' = I - \kappa'[1 - \cos^2 v\sin^2(u + I)][1 + \tan 2I\tan(u + I)],$$

where $\kappa' = (a^2 - c^2)/2c(1 + c^2)$, and $I = \cot^{-1} c$ is Brewster's angle. This formula represents very accurately the results of Schmidt's observations with the extraordinary ray, the difference between the observed and the calculated values of I' being never greater than $18'$, and usually only a few minutes.

Influence of a Layer of Transition.

We must now consider to what extent these formulae are changed when we regard the transition into the crystal to be gradual, and not abrupt. The general character of the results might be predicted from the analogy of the behaviour of isotropic media, but without more minute examination it is not easy to decide with certainty to what extent the crystalline structure

modifies the results previously obtained. As nearly all the experiments have been made with uniaxal crystals cut either parallel or perpendicular to the optic axis, we shall consider these two cases more particularly. Just as before it is convenient to deal first with a uniradial system, and if the incident displacement be not in one of the uniradial directions to resolve it into components along these directions.

If A be the end of the optic axis, and AM the perpendicular from A on the wave front (Fig. 65), then the uniradial directions are along OM and OM', the latter being at right angles to OM. Of these OM corresponds to the ordinary and OM' to the extraordinary ray. Whenever, then, the optic axis lies in the plane of incidence, which happens among other cases when the crystal is cut at right angles to the axis, the uniradial directions are in the plane of incidence and at right angles thereto. Under such circumstances the argument used in the case of isotropic media may be applied to crystals with very slight modifications*. It will be sufficient to indicate the procedure for a crystal cut at right angles to its optic axis. As usual we derive everything from the Principle of Action, taking $W = \frac{1}{2}\int \Phi d\tau$, where in the air $2\Phi = f^2 + g^2 + h^2$ (the velocity in air being taken as unity), and in the crystal $2\Phi = a^2 f^2 + c^2(g^2 + h^2)$, while in the layer of transition $2\Phi = A^2 f^2 + C^2(g^2 + h^2)$. Here A and C are functions of x, and the continuity of the elastic properties of the medium requires that $A = 1$ and $C = 1$ when $x = 0$, and $A = a$ and $C = c$ when $x = d$, where d is, as before, the thickness of the layer.

When the displacement is perpendicular to the plane of incidence we have $\xi = \eta = 0$ and $\zeta = ue^{ip(t-\nu y)}$,

$$f = \frac{\partial \zeta}{\partial y},\ g = -\frac{\partial \zeta}{\partial x},\ h = 0,$$

while the boundary conditions are satisfied if ζ and $\frac{\partial \zeta}{\partial x}$ are continuous, i.e. if u and $\frac{du}{dx}$ are continuous. The dynamical equation

* Of course any incident displacement can be resolved in directions parallel and perpendicular to the plane of incidence; but unless these are uniradial directions each component will give rise to *two* waves in the crystal, and this will greatly complicate the discussion.

is $\quad \ddot{\xi} = \frac{\partial}{\partial y}(A^2 f) - \frac{\partial}{\partial x}(B^2 g)$, i.e. $-p^2 u = -A^2 p^2 \nu^2 u + \frac{d}{dx}\left(C^2 \frac{du}{dx}\right)$,

or
$$\frac{d}{dx_1}\left(C^2 \frac{du}{dx_1}\right) + d_1^2 (1 - A^2\nu^2) u = 0.$$

Solving this by successive approximations, regarding d_1^2 as small, we get as the first approximation

$$u = A_1 + MB_1/C^2 \text{ where } M = C^2 \int_0^{x_1} (1/C^2)\, dx_1$$

and as the next approximation $u = A_1 + MB_1/C^2 + d_1^2 . v$, where

$$\frac{d}{dx_1}\left(C^2 \frac{dv}{dx_1}\right) + (1 - A^2\nu^2)(A_1 + MB_1/C^2) = 0.$$

On integrating this gives

$$dv/dx_1 = -A_1 x/C^2 - B_1 M_1/C^4 + A_1 \nu^2 M_3 + B_1 \nu^2 M_4/C^2,$$

and $v = -A_1(xM - M_1)/C^2 - B_1 M_5/C^4 + A_1 \nu^2 M_6 + B_1 \nu^2 M_7/C^2$, where

$$M_1/C^2 = \int_0^{x_1} (M/C^2)\, dx_1\,; \quad C^2 M_3 = \int_0^{x_1} A^2 dx_1\,;$$

$$M_4 = \int_0^{x_1} (A^2/C^2)\, dx_1\,; \quad M_5 = C^4 \int_0^{x_1} (M_1/C^4)\, dx_1\,;$$

$$M_6 = \int_0^{x_1} M_3 dx_1\,; \qquad M_7 = C^2 \int_0^{x_1} (M_4/C^2)\, dx_1.$$

When $x_1 = 0$ we have $u = A_1$, $du/dx_1 = B_1$ and when $x_1 = 1$ we have $u = A_1 + EB_1/c^2 + d_1^2 [A \{(H - E)/c^2 + \nu^2 L\} + (B_1/c^2)\{\nu^2 N - K/c^2\}]$, and $du/dx_1 = B_1/c^2 + d_1^2 [A\{\nu^2 F - 1/c^2\} + (B_1/c^2)\{\nu^2 J - H/c^2\}]$, exactly, as before, p. 65, except that μ_1 is replaced by $1/c$, and the constants E, F, H, etc. are altered. Hence all the results previously obtained apply to this case, the refractive index that occurs in the formulae being that for the ordinary ray.

Similarly, when the displacement is parallel to the plane of incidence, we have $\xi' = \eta' = 0$, $\zeta' = ue^{ip(t-\nu y)}$;

$$\xi = \frac{\partial \zeta'}{\partial y}, \quad \eta = -\frac{\partial \zeta'}{\partial x}, \quad \zeta = 0\,; \quad f = 0, \quad g = 0, \quad h = -\nabla^2 \zeta'.$$

The boundary conditions are satisfied if u and du/dx are continuous, and the dynamical equations give $\ddot{\zeta}' - C^2 \nabla^2 \zeta' = 0$, i.e. $\frac{d^2 u}{dx^2} + p^2 (1/C^2 - \nu^2) u = 0$. This equation is of the same form as that

obtained when dealing with isotropic media (p. 69), and its discussion yields results identical in form with those obtained in that case.

Turning now to the consideration of a crystal cut parallel to its optic axis, we shall suppose that the axis of the crystal makes an angle l with the plane of incidence. We then have

$$2\Phi = a^2 f'^2 + c^2 (g'^2 + h'^2) = (a^2 - c^2) f'^2 + c^2 (f'^2 + g'^2 + h'^2).$$

Hence on referring Φ to the coordinate axes instead of the principal axes of the crystal, we get

$$\begin{aligned} 2\Phi &= (a^2 - c^2)(g \cos l + h \sin l)^2 + c^2 (f^2 + g^2 + h^2) \\ &= Af^2 + Bg^2 + Ch^2 + 2Fgh, \end{aligned}$$

where $\quad A = c^2;\ B = c^2 + (a^2 - c^2)\cos^2 l;\ C = c^2 + (a^2 - c^2)\sin^2 l;$

$$F = (a^2 - c^2)\sin l \cos l;\ BC - F^2 = a^2 c^2.$$

The Principle of Action requires $Ch + Fg$ and $Bg + Fh$ to be continuous at the interface and gives the dynamical equations in the form :—

$$\ddot{\xi} = -\frac{\partial}{\partial y}(Fg + Ch);\quad \ddot{\eta} = \frac{\partial}{\partial x}(Fg + Ch);$$

$$\ddot{\zeta} = \frac{\partial}{\partial y}(Af) - \frac{\partial}{\partial x}(Bg + Fh).$$

In the air we have

$$(\xi, \eta, \zeta) = (L, M, N) A e^{ip(t - x\cos\phi - y\sin\phi)} \quad \text{(incident wave)}$$
$$+ (L', M', N') A' e^{ip(t + x\cos\phi - y\sin\phi)} \quad \text{(reflected wave)}.$$

In the crystal

$$(\xi, \eta, \zeta) = (L_1, M_1, N_1) A_1 e^{ip\{t - (x\cos\phi_1 + y\sin\phi_1)/v_1\}} \quad \text{(refracted wave)},$$

there being only one refracted wave, as the incidence is uniradial by hypothesis. In the layer we have $(\xi, \eta, \zeta) = (u, v, w)\, e^{ip(t - \nu y)}$, where u, v, w are functions of x only. Since

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0,\ \text{ we have } \frac{du}{dx} - ip\nu v = 0,$$

and therefore $\quad v = -\dfrac{i}{\nu p}\dfrac{du}{dx}.$

Also $f = \dfrac{\partial \zeta}{\partial y} = - i\nu p w e^{ip(t-\nu y)}$; $g = -\dfrac{\partial \zeta}{\partial x} = -\dfrac{dw}{dx} e^{ip(t-\nu y)}$;

$$h = \frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y} = -\frac{i}{\nu p}\left[\frac{d^2u}{dx^2} - \nu^2 p^2 u\right] e^{ip(t-\nu y)}.$$

Substituting in the dynamical equations we get

$$C\frac{d^2u}{dx^2} + p^2(1 - \nu^2 C)u = F\frac{dw_2}{dx},$$

and $$\frac{d}{dx}\left[F\left(\frac{d^2u}{dx^2} - \nu^2 p^2 u\right)\right] = \frac{d}{dx}\left(B\frac{dw_2}{dx}\right) + p^2(1 - \nu^2 A)w_2,$$

where $w_2 = ip\nu w$.

Putting $x/d = x_1$, $d_1 = pd$, as before, and $w_1 = d_1 w_2/p$, these equations take the forms

$$C\frac{d^2u}{dx_1^2} + d_1^2(1 - \nu^2 C)u = F\frac{dw_1}{dx_1},$$

and $$\frac{d}{dx_1}\left[F\left(\frac{d^2u}{dx_1^2} - d_1^2\nu^2 u\right)\right] = \frac{d}{dx_1}\left(B\frac{dw_1}{dx_1}\right) + d_1^2(1 - A\nu^2)w_1.$$

These equations are to be solved by successive approximations. Neglecting d_1^2 we get

$$C\frac{d^2u}{dx_1^2} = F\frac{dw_1}{dx_1},$$

and $$\frac{d}{dx_1}\left[F\left(\frac{d^2u}{dx_1^2}\right)\right] = \frac{d}{dx_1}\left(B\frac{dw_1}{dx_1}\right).$$

The second of these gives

$$B\frac{dw_1}{dx_1} = F\frac{d^2u}{dx_1^2} + \gamma,$$

where γ is a constant.

Substituting in the first we get

$$\frac{d^2u}{dx_1^2} = \frac{\gamma F}{BC - F^2} = \gamma H \text{ say};$$

whence we derive

$$\frac{du}{dx_1} = \beta + \gamma\int_0^{x_1} H\,dx_1 = \beta + \gamma I;$$

$$u = \alpha + \beta x_1 + \gamma J, \text{ where } J = \int_0^{x_1} I\,dx_1;$$

and $$\frac{dw_1}{dx_1} = \frac{\gamma C}{BC - F^2} = \gamma K;\ w_1 = \alpha_1 + \gamma\int_0^{x_1} K\,dx_1 = \alpha_1 + \gamma\,.\,P.$$

As the next approximation we put

$$u = \alpha + \beta x_1 + \gamma J + d_1^2 . u'; \quad w_1 = \alpha_1 + \gamma . P + d_1^2 w';$$

and the equations for u' and w' are

$$C\frac{d^2u'}{dx_1^2} + (1 - C\nu^2)(\alpha + \beta x_1 + \gamma J) = F\frac{dw'}{dx_1},$$

and

$$\frac{d}{dx_1}\left[F\frac{d^2u'}{dx_1^2}\nu^2 F(\alpha + \beta x_1 + \gamma J)\right]$$

$$= \frac{d}{dx_1}\left(B\frac{dw'}{dx_1}\right) + (1 - A\nu^2)(\alpha_1 + \gamma P).$$

Before solving these equations we may use the boundary conditions at $x = 0$ to determine the relative magnitudes of the constants α, β, γ, and α_1. Of the five boundary conditions only four are independent, and these we may take to require the continuity of ξ, η, ζ and $Bg + Fh$. The continuity of ξ, η, ζ involves that of u, v, w, and so of $u, \dfrac{du}{dx}$, and w; while that of $Bg + Fh$ involves the continuity of

$$\frac{p}{d_1} B\frac{dw_1}{dx} - F\left(\frac{d^2u}{dx^2} - \nu^2 p^2 u\right).$$

When $x_1 = 0$ we have

$$u = \alpha, \quad w_1 = \alpha_1, \quad du/dx_1 = \beta, \quad dw_1/dx_1 = \gamma, \quad B = 1, \quad F = 0.$$

Hence the boundary conditions at $x_1 = 0$ give

$$\alpha = AL + A'L'; \quad \beta = i\nu d_1(AM + A'M');$$

$$\gamma = \nu d_1^2 \cos\phi (AN - A'N'); \quad \alpha_1 = i\nu d_1(AN + A'N').$$

As β and α_1 are of the first order in d_1 and γ is of the second order we may simplify the equations for u' and w' by omitting terms containing these constants, since u' and w' are multiplied by d_1^2 in the equations for u and w_1, and we are neglecting powers of d_1 above the second. In this way we get

$$C\frac{d^2u'}{dx_1^2} + \alpha(1 - C\nu^2) = F\frac{dw'}{dx_1},$$

and

$$\frac{d}{dx_1}\left[F\frac{d^2u'}{dx_1^2} - \alpha\nu^2 F\right] = \frac{d}{dx_1}\left(B\frac{dw'}{dx_1}\right).$$

Taking the simplest solutions possible we get

$$\frac{d^2u'}{dx_1^2} = \alpha\left[\nu^2 - \frac{B}{BC - F^2}\right] = \alpha\,[\nu^2 - P],$$

$$\frac{du'}{dx_1} = \alpha\left[\nu^2 x_1 - \int_0^{x_1} P dx_1\right] = \alpha\,(\nu x_1 - Q),$$

$$u' = \alpha\left[\frac{\nu^2 x_1^2}{2} - \int_0^{x_1} Q dx_1\right] = \alpha\,[\nu^2 x_1^2/2 - R]$$

$$\frac{dw'}{dx_1} = -\alpha H, \quad w' = -\alpha I.$$

Hence when $x_1 = 1$ we have

$$u = \alpha + \beta + \gamma J + \alpha d_1^2(\nu^2/2 - R);$$

$$\frac{du}{dx_1} = \beta + \gamma I + \alpha d_1^2(\nu^2 - Q); \quad \frac{d^2u}{dx_1^2} = \gamma H + \alpha d_1^2(\nu^2 - P);$$

$$w_1 = \alpha_1 + \gamma P - \alpha d_1^2 I; \quad \frac{dw_1}{dx_1} = \gamma K - \alpha d_1^2 H;$$

in which all the integrals are made definite by putting $x_1 = 1$ at the upper limit, e.g.,

$$I = \int_0^1 \frac{F dx_1}{BC - F^2}.$$

If now we substitute these expressions in the boundary conditions at $x_1 = 1$, and as the first approximation neglect terms in d_1 of a higher order than the first, then, since $BK - FH = 1$ and $BH - FP = 0$, we get the following equations:—

$$A_1 L_1 = AL + A'L' + i\nu d_1(AM + A'M'),$$

$$A_1 M_1 = AM + A'M' - id_1[I\cos\phi\,(AN - A'N') + (AL + A'L')(\nu^2 - Q)/\nu],$$

$$A_1 N_1 = AN + A'N' - id_1[P\cos\phi\,(AN - A'N') - (AL + A'L')\,I/\nu],$$

$$\sin\phi\,[BA_1N_1\cot\phi_1 + FA_1L_1\operatorname{cosec}^2\phi_1] = \cos\phi\,(AN - A'N') + id_1\sin^2\phi\,(AM + A'M').$$

Expressing L, M, N in terms of θ and ϕ as on p. 159, and putting $A\cos\theta = x$ and $A\sin\theta = y$ for brevity, we get

$$x_1\sin\phi_1 = (x + x')\sin\phi - id_1\sin\phi\cos\phi\,(x - x'),$$

$$x_1\cos\phi_1 = (x - x')\cos\phi + id_1[I\cos\phi\,(y + y') + (\sin^2\phi - Q)(x + x')],$$

$$y_1 = y - y' - id_1[P\cos\phi\,(y + y') - I(x + x')],$$

$$(By_1\cos\phi_1 + Fx_1)\sin\phi/\sin\phi_1 = \cos\phi\,(y + y') - id_1\sin^2\phi\cos\phi\,(x - x').$$

Eliminating x_1 and y_1 we get

$$x'\left[\sin(\phi+\phi_1)+id_1\{\sin\phi\cos\phi\cos\phi_1-\sin\phi_1(\sin^2\phi-Q)\}\right]$$
$$-y'\,id_1 I\cos\phi\sin\phi_1$$
$$=-x\left[\sin(\phi-\phi_1)-id_1\{\sin\phi\cos\phi\cos\phi_1+\sin\phi_1(\sin^2\phi-Q)\}\right]$$
$$+y\,id_1 I\cos\phi\sin\phi_1,$$

and
$$x'\left[F\sin^2\phi-id_1\{\sin^2\phi\sin^2\phi_1\cos\phi-F\sin^2\phi\cos\phi\right.$$
$$\left.-BI\sin\phi\sin\phi_1\cos\phi_1\}\right]$$
$$-y'\left[\sin\phi_1(\sin\phi_1\cos\phi+B\cos\phi_1\sin\phi)\right.$$
$$\left.+id_1 BP\sin\phi\cos\phi\sin\phi_1\cos\phi_1\right]$$
$$=-x\left[F\sin^2\phi+id_1\{\sin^2\phi\sin^2\phi_1\cos\phi-F\sin^2\phi\cos\phi\right.$$
$$\left.+BI\sin\phi\sin\phi_1\cos\phi_1\}\right]$$
$$+y\left[\sin\phi_1(\sin\phi_1\cos\phi-B\cos\phi_1\sin\phi)\right.$$
$$\left.+id_1 BP\sin\phi\cos\phi\sin\phi_1\cos\phi_1\right].$$

From these we get

$$y'/x'=(p+id_1p_1)/(q+id_1q_1)=\epsilon e^{i\Delta},$$

where ϵ is the ratio of the amplitudes, and Δ the difference of phase between the components parallel and perpendicular to the plane of incidence. As d_1 is very small we have $y'/x'=p/q$ very nearly, unless p or q is very small. Hence as a rule ϵ is very nearly the same as if there were no layer and the difference of phase Δ is very nearly zero or π. This, however, will not be the case in the neighbourhood of the uniradial "polarising angle," for p is zero at this angle.

When $p=0$ we have $\epsilon e^{i\Delta}=id_1p_1/q$, so that the coefficient of ellipticity is $\epsilon_1=\pm d_1p_1/q$, and the difference of phase is $\pi/2$ or $-\pi/2$ according to the sign of p_1/q. This, then, is the Principal Incidence and in its neighbourhood we have, approximately, $\tan\Delta=d_1p_1/p=\epsilon_1 q/p$. With the aid of the above equations these formulae give

$$\epsilon_1=-d_1\frac{F\sin 2\phi\sin\phi_1(\sin^2\phi+BI)+m\tan\theta}{\sin(\phi-\phi_1)(\sin\phi_1\cos\phi+B\cos\phi_1\sin\phi)},$$

where

$$m=IF\sin^2\phi\cos\phi-\sin\phi\cos\phi\cos\phi_1\{BP\sin(\phi+\phi_1)$$
$$+\sin\phi_1\cos\phi-B\cos\phi_1\sin\phi\}$$
$$+\sin\phi_1(\sin^2\phi-Q)(\sin\phi_1\cos\phi-B\cos\phi_1\sin\phi),$$

and

$$\tan \Delta = \epsilon_1 \frac{\sin(\phi - \phi_1)(\sin\phi_1 \cos\phi + B\cos\phi_1 \sin\phi)}{\tan\theta \sin(\phi+\phi_1)(\sin\phi_1\cos\phi - B\cos\phi_1\sin\phi) - F\sin 2\phi\sin\phi}.$$

At the Principal Incidence, $\tan\Delta$ is infinite and

$$\tan\theta = F\sin 2\phi \sin\phi/\sin(\phi+\phi_1)(\sin\phi_1\cos\phi - B\cos\phi_1\sin\phi).$$

The reflection is positive or negative according as Δ varies from 0 to $-\pi$, i.e. according as p_1/q is negative or positive. Hence the reflection is positive or negative according as

$$F\sin 2\phi \sin\phi_1 (\sin^2\phi + BI) + m\tan\theta$$

is positive or negative, i.e. according as

$$F\sin 2\phi\,[\sin\phi_1(\sin^2\phi + BI)$$
$$+ m\sin\phi/\sin(\phi+\phi_1)(\sin\phi_1\cos\phi - B\cos\phi_1\sin\phi)]$$

is positive or negative. Since $F = (a^2 - c^2)\sin l\cos l$, the sign of F will vary with the orientation of the crystal and the reflection will change from positive to negative, as is found experimentally to be the case.

For some purposes it is convenient to express x' and y' in terms of x_1 and y_1, the angle θ_1 being readily given in terms of ϕ_1 and l by the formulae on p. 168. Eliminating x and y from the equations on p. 184, we get

$$x'[2\cos\phi + id_1(2\cos^2\phi - \sin^2\phi + Q)] = x_1[-\sin(\phi-\phi_1)/\sin\phi$$
$$+ id_1\{(1-Q)\sin\phi_1/\sin\phi + IF\sin\phi/\sin\phi_1\}]$$
$$+ id_1 y_1 IB\sin\phi\cos\phi/\sin\phi_1,$$

and

$$x'[2id_1\sin^2\phi] + y'[1 + id_1 P\cos\phi] = x_1[F\sin\phi/\sin\phi_1\cos\phi$$
$$- id_1\{FP\sin\phi/\sin\phi_1 - \sin\phi\sin\phi_1 - I\sin\phi_1/\sin\phi\}]$$
$$- y_1[(\sin\phi_1\cos\phi - B\sin\phi\cos\phi_1)/\sin\phi_1\cos\phi$$
$$+ id_1 BP\sin\phi\cos\phi_1/\sin\phi_1].$$

From these we derive

$$y'/x' = (p + id_1p_1)/(q + id_1q_1) = \epsilon e^{i\Delta}$$

as before, where

$$\epsilon_1 = d_1p_1/q = -m'd_1/\sin\phi_1\cos\phi\sin(\phi-\phi_1),$$

$$m' = 2\sin^3\phi\cos\phi\sin\phi_1\cos\phi_1 - 2FP\sin^2\phi\cos^2\phi + 2I\sin^2\phi_1\cos^2\phi$$
$$+ F\sin^2\phi(2\cos^2\phi - \sin^2\phi + Q) - \sin\phi\tan\theta_1[2B\sin\phi\cos^2\phi\cos\phi_1$$
$$+ (2\cos^2\phi - \sin^2\phi + Q)(\sin\phi_1\cos\phi - B\cos\phi_1\sin\phi)],$$

and $\tan\Delta = \epsilon_1 q/p$

$$= \epsilon_1 \frac{\sin\phi_1 \sin(\phi - \phi_1)}{2\sin\phi\,[(\sin\phi_1\cos\phi - B\cos\phi_1\sin\phi)\tan\theta_1 - F\sin\phi]}.$$

This formula for Δ is somewhat simpler than the equivalent one found above in terms of $\tan\theta$*. The following tables give the values of Δ in the neighbourhood of the uniradial polarising angles for Iceland spar at different orientations and compare the results of theory with the experimental measurements made by Schmidt.

Ordinary ray ($l = 78°\ 52'$).

ϕ......	57° 23′	58° 23′	59° 23′	60° 23′	62° 23′
$-\Delta$ (theory)	0·06	0·16	0·73	0·90	0·94
$-\Delta$ (exp.)	0·06	0·16	0·72	0·90	0·94

Extraordinary ray ($l = -44°\ 39'$).

ϕ......	56° 32′	58° 32′	59° 32′	60° 32′	62° 32′
$-\Delta$ (theory)	0·04	0·11	0·63	0·90	0·96
$-\Delta$ (exp.)	0·04	0·12	0·62	0·90	0·96

These results are shown in Fig. 69.

If we wish to discuss the refracted ray, we eliminate x' and y' from the equations on p. 184 and so get

$$x_1[\sin(\phi+\phi_1)/\sin\phi - id_1(1-Q)\sin\phi_1/\sin\phi] + id_1 I\cos\phi\, y_1$$
$$= 2x\cos\phi\,[1 - id_1\cos\phi] + 2id_1 I\cos\phi\,.\,y,$$

and $$x_1[F\sin\phi/\sin\phi_1 - id_1\sin\phi_1\sin\phi\,(\sin\phi + I\,\mathrm{cosec}\,\phi)]$$
$$+ y_1[s_1\,\mathrm{cosec}\,\phi_1 - id_1 P\cos^2\phi]$$
$$= -2id_1 x\sin^2\phi\cos\phi + 2y(1 - id_1 P\cos\phi),$$

where $$s_1 = \cos\phi + B\sin\phi\cot\phi_1.$$

* At the Principal Incidence we have

$$\cot\theta_1 = (\sin\phi_1\cos\phi - B\cos\phi_1\sin\phi)/F\sin\phi$$

which with $\cot\theta_1 = -\cos\phi_1\cot l$ for the ordinary ray, and $\cot\theta_1 = \sec\phi_1\tan l$ for the extraordinary ray, enables us to calculate the uniradial polarising angles as on p. 176 above. The velocities of the ordinary and extraordinary waves are $\sqrt{A}$ and $\sqrt{C+(B-A)\cos^2\phi_1}$ respectively.

From these we derive

$$y_1/x_1 = \frac{s_1 \sin(\phi + \phi_1) \tan\theta - 2F \sin\phi \cos\phi + id_1 m'' \sin\phi_1}{2s_1 \cos\phi - 2id_1 \cos\phi\,(s_1 \cos\phi - IB \sin\phi \cos\phi_1 \tan\theta)},$$

$$= \epsilon' e^{i\Delta'} \text{ say,}$$

where

$$m'' = 2\cos^2\phi[\sin\phi_1 \sin\phi + I \sin\phi_1 \operatorname{cosec}\phi + F \sin\phi \operatorname{cosec}\phi_1]$$
$$- \tan\theta\,[2IF \sin\phi \cos\phi \operatorname{cosec}\phi_1 + s_1(1 - Q) \operatorname{cosec}\phi$$
$$+ P \cos^2\phi \sin(\phi + \phi_1) \operatorname{cosec}\phi].$$

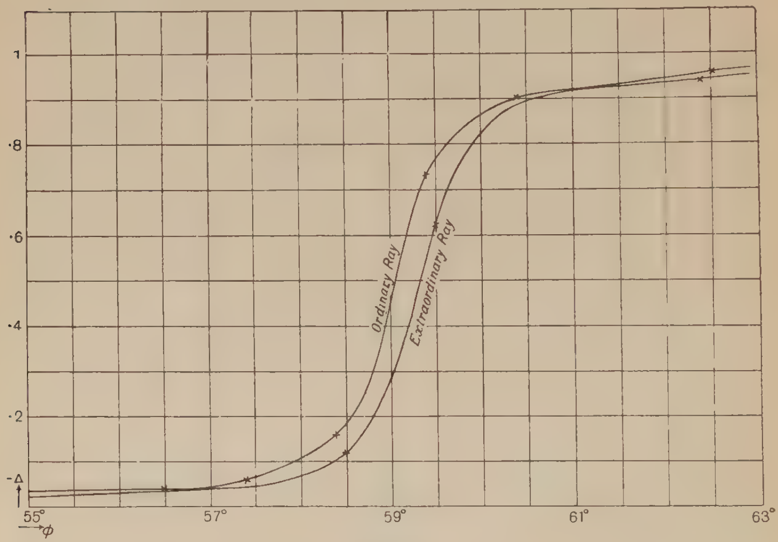

Fig. 69.

The refracted wave is elliptically polarised, the major axis of the elliptic orbit making an angle ψ_1 with the plane of incidence, where $\tan 2\psi_1 = \tan 2\alpha_1 \cos\Delta'$ and $\tan\alpha_1 = \epsilon'$ (cf. p. 84). As Δ' is small, the ellipse is approximately a straight line, and the light is very nearly plane polarised in the azimuth ψ_1. If the transition were abrupt, the light would be plane polarised in an azimuth θ_1 where

$$\tan\theta_1 = [s_1 \sin(\phi + \phi_1) \tan\theta - 2F \sin\phi \cos\phi]/2s_1 \cos\phi,$$

so that the effect of the layer is to turn the plane of polarisation through an angle $\psi_1 - \theta_1$.

Internal Reflection.

Returning to the ideal case of an abrupt transition, we shall next consider the problem of internal reflection at the second surface of a crystal. This problem may, as MacCullagh pointed out, be reduced in certain cases to one that has already been solved. Thus suppose that we have a crystalline plate with parallel faces and that we wish to determine the amplitude and azimuth of the reflected and refracted displacements corresponding to an internal ray R_1. This ray will, in general, give rise to two

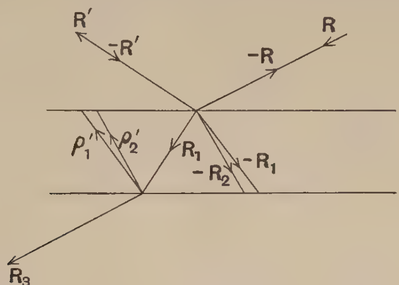

Fig. 70.

reflected rays ρ_1' and ρ_2' and a refracted ray R_3. The ray R_1 could be produced by an incident ray R in a uniradial direction, and corresponding to this we should also have a reflected ray R'. The amplitude and direction of the displacement in R_1 being given, we can determine those in R and R' very simply by MacCullagh's theorem of the polar plane (p. 163). The directions of the displacements are the intersections of the incident and reflected waves with the polar plane of R_1 and their magnitudes are given by the parallelogram law, since they are the components of a vector of which the displacement in R_1 is the resultant. If now the direction of the displacement in R' were reversed and the ray turned back so as to become $-R_1'$, it would give rise to a reflected ray $-R$ and two refracted rays $-R_1'$ and $-R_2'$, and it follows from the dynamical principle of reversion that ρ_1' and ρ_2' are the same as $-R_1'$ and $-R_2'$ except for the change of sense. The displacements in $-R_1'$ and $-R_2'$ can be determined by methods already discussed, so that those in ρ_1' and ρ_2' can be obtained. The dis-

placement in R_3 will be equal and parallel to the resultant of those in R and $-R$.

The method indicated above gives the complete solution of the problem of partial reflection; but if we wish to make a numerical measure of the amplitudes, and in particular if we wish to deal with the case of *total* reflection, it is better to abandon the geometrical method of MacCullagh and to proceed straightforwardly by writing down the boundary conditions exactly as on p. 160 for the special problem in hand. Thus, let the incident ray 1 give rise to two reflected rays 2, 3 and a refracted ray 4 (Fig. 71). As before, denote the amplitude of a displacement

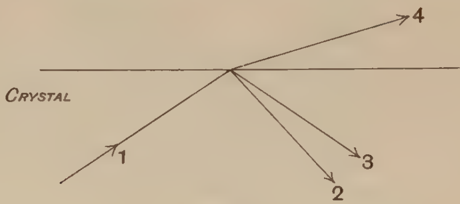

Fig. 71.

by A and its direction cosines by L, M, N. For brevity we shall use the symbol S to denote the quantity

$$\sin\theta \sin\phi \cos\phi + \tan\chi \sin^2\phi$$

(see p. 160). The boundary conditions then give

$$A_1L_1 + A_2L_2 + A_3L_3 = A_4L_4 = A_4 \cos\theta_4 \sin\phi_4;$$

$$A_1M_1 + A_2M_2 + A_3M_3 = A_4M_4 = -A_4 \cos\theta_4 \cos\phi_4;$$

$$A_1N_1 + A_2N_2 + A_3N_3 = A_4N_4 = A_4 \sin\theta_4;$$

$$A_1S_1 + A_2S_2 + A_3S_3 = A_4S_4 = A_4 \sin\theta_4 \sin\phi_4 \cos\phi_4.$$

These are to be coupled with

$$\sin\phi_1/v_1 = \sin\phi_2/v_2 = \sin\phi_3/v_3 = \sin\phi_4/v_4 = \sin\phi_4; \text{ taking } v_4 = 1.$$

These last equations, together with the properties of the Index Surface, determine ϕ_2, ϕ_3 and ϕ_4 in terms of ϕ_1 and then the equations representing the boundary conditions are just sufficient to determine A_2, A_3, A_4 and θ_4 in terms of A_1 and θ_1. If we wish to consider particularly the reflected rays we eliminate $A_4 \cos\theta_4$ from

the first two equations, and $A_4 \sin\theta_4$ from the last two, and so obtain

$$A_1(L_1\cot\phi_4 + M_1) + A_2(L_2\cot\phi_4 + M_2) + A_3(L_3\cot\phi_4 + M_3) = 0$$

and

$$A_1(N_1\sin\phi_4\cos\phi_4 - S_1) + A_2(N_2\sin\phi_4\cos\phi_4 - S_2) \\ + A_3(N_3\sin\phi_4\cos\phi_4 - S_3) = 0,$$

whence $$A_1/\Delta_{23} = A_2/\Delta_{31} = A_3/\Delta_{12},$$

where $$\Delta_{pq} = \begin{vmatrix} L_p\cot\phi_4 + M_p & L_q\cot\phi_4 + M_q \\ N_p\sin\phi_4\cos\phi_4 - S_p & N_q\sin\phi_4\cos\phi_4 - S_q \end{vmatrix}.$$

In case of partial reflection ϕ_4 is real, and in this case the Δ's are all real. Thus A_2 and A_3 are real, and the change of phase produced by reflection is zero or π, as is found experimentally to be the case when the layer of transition at the surface of the crystal is negligible. If, however, ϕ_4 be not real, then A_2 and A_3 are in general complex, so that there will be a change of phase on reflection. If the outer medium were absorptive then, as we shall see later, v_4 would be complex, so that $\sin\phi_4$ would be complex, and a change of phase would always be expected. If the outer medium be transparent, $\sin\phi_4$ must be real, but it may be greater than unity (as will happen when $\sin\phi_1 > v_1$), so that $\cos\phi_4$ and $\cot\phi_4$ will be purely imaginary. Under such circumstances ϕ_4 is imaginary and the reflection is *total*. There is no difficulty in obtaining formulae for the amplitude and phase of each of the reflected rays.

Putting $\sin\phi_4 = q = \cosh x$, we have

$$\begin{aligned}\Delta_{pq} &= (iL_p\tanh x + M_p)(iN_q\sinh x\cosh x - S_q) \\ &\qquad - (iL_q\tanh x + M_q)(iN_p\sinh x\cosh x - S_p) \\ &= (N_pL_q - L_pN_q)\sinh^2 x + (S_pM_q - M_pS_q) \\ &\qquad + i\tanh x\,[(S_pL_q - L_pS_q) + (M_pN_q - N_pM_q)\cosh^2 x] \\ &= D_{pq}\,e^{i\theta_{pq}}.\end{aligned}$$

Thus

$$A_2/A_1 = \Delta_{31}/\Delta_{23} = (D_{31}/D_{23})\,e^{i(\theta_{31}-\theta_{23})}, \text{ and } A_3/A_1 = (D_{12}/D_{23})\,e^{i(\theta_{12}-\theta_{23})}.$$

The change of phase in the two cases is $\theta_{31} - \theta_{23}$ and $\theta_{12} - \theta_{23}$, and the difference of phase between the two reflected rays is $\theta_{12} - \theta_{31}$.

It thus appears that all that is unknown may be expressed in terms of L, M, N, and S. These quantities, in turn, depend on θ, ϕ, and χ, which are given by the theorems of the last chapter. The resulting formulae for the amplitude and phase in the reflected waves are very complicated in the general case. There is, however, a considerable simplification when the crystal is uniaxal, and a still further lightening of the labours of calculation from the fact that in practice the difference between a and c is always small.

We shall take the same axes and adopt the same notation as on p. 167, and shall suppose the incident ray 1 to be ordinary, and the reflected rays 2 and 3 to be ordinary and extraordinary respectively. We then have

$$\sin\phi_1/c = \sin\phi_2/c = \sin\phi_3/v_3 = \sin\phi_4 = q = \cosh x\,;$$

$$v_3^2 = a^2 - (a^2 - c^2)\cos^2\omega_3\,; \quad \tan\chi_3 = (a^2 - c^2)\sin\omega_3\cos\omega_3/v_3^2\,;$$

$$\cos\omega_3 = \cos v\cos(\phi_3 - u)\,; \quad \chi_1 = \chi_2 = 0\,;$$

$$\sin(\phi_1 \sim u) = -\cot\theta_1\tan v\,; \quad \sin(\phi_2 \sim u) = \cot\theta_2\tan v\,;$$

$$\sin(\phi_3 \sim u) = -\tan\theta_3\tan v.$$

From these we get

$$\phi_2 = \pi - \phi_1\,; \quad \sin^2\phi_3 - \sin^2\phi_2 = q^2(v_3^2 - a^2) = -q^2(a^2 - c^2)\cos^2\omega_3\,;$$

so that $\qquad \phi_3 - \phi_2 = \sin(\phi_3 - \phi_2)$ approximately

$$= q^2(a^2 - c^2)\cos^2\omega_3/\sin 2\phi_1 = y,$$

where y is small. There are two special cases worth dealing with: (1) when the face of the crystal is perpendicular to the optic axis, and (2) when it is parallel to that axis.

In the first case we have $u = v = 0, \quad \theta_3 = -\pi/2$,

$$\omega_3 = \phi_3 = \pi - \phi_1 + y\,; \quad y = (q^2/2)(a^2 - c^2)\cot\phi_1\,;$$

$$\tan\chi_3 = -q^2(a^2 - c^2)\cot\phi_1 = -2y.$$

The sign to be attributed to $\tan\chi_3$ in the formula for S_3 has not yet been determined. It is fixed by the convention of signs indicated in Fig. 61 and the note on p. 158 that $\tan\chi$ is taken positively when the wave normal lies between the ray and the positive direction of the curl. In the present case, if $\tan\chi_3$ be taken positively, the extraordinary ray is nearer the optic axis than the ordinary ray, which Huyghens' construction shows to be the case with a positive crystal. Hence $\tan\chi_3$ must be taken positively or

negatively according as the crystal is positive or negative. The equations on which the solution of the problem depends may now be written down. We have $L_1 = \sin\phi_1 = L_2$; $M_1 = -\cos\phi_1 = -M_2$; $N_1 = N_2 = S_1 = S_2 = L_3 = M_3 = 0$; $N_3 = 1$; $S_3 = -\sin\phi_1\cos\phi_1 + z$, where $z = y$ if the crystal be positive, and $z = y\,(1 - 4\sin^2\phi_1)$ if the crystal be negative. These make $\Delta_{12} = 0$, so that $A_2 = 0$, and there is no extraordinary ray. If the incident ray were extraordinary we should take $\theta_1 = \theta_3 = 0$, $\theta_2 = \pi/2$, and this would make $\Delta_{31} = 0$, and so $A_2 = 0$. Hence with a crystal cut at right angles to the optic axis an ordinary ray gives rise to an ordinary ray and an extraordinary ray to an extraordinary one, as might have been anticipated from the perfect symmetry with respect to the axis.

Returning to the case when the incident ray is ordinary we get

$$\tan\theta_{31} = \tanh x\cot\phi_1 + z\tanh x/\sin^2\phi_1\,(q^2 - \sin^2\phi_1)$$
$$= \tanh x\,[\cot\phi_1 + z/(\mu^2 - 1)\sin^4\phi_1], \quad \text{where } \mu = 1/a;$$

and

$$\tan\theta_{23} = \tanh x\,[\cot\phi_1\,(\mu^2+1)\sin^2\phi_1/\{2 - (\mu^2+1)\sin^2\phi_1\}$$
$$+ (\mu^2 - 1)\,z/\{2 - (\mu^2+1)\sin^2\phi_1\}^2],$$

which determines the change of phase. It is easy to verify that $D_{31} = D_{23}$, so that $A_2 = A_1$, as should be the case with total reflection, when there is no loss of light.

We shall next consider the case when the crystal is cut parallel to the optic axis. To deal with this we must put $u = \pi/2$ in the general formulae of p. 192. This gives

$$\cos\omega_3 = \cos v\sin\phi_3; \quad y = (q^2/2)\,(a^2 - c^2)\cos^2 v\tan\phi_1; \quad \theta_2 = -\theta_1,$$

as is obvious geometrically;

$$\tan\theta_3 = -\cot\theta_2\cos\phi_3\sec\phi_2 = -\cot\theta_2\,(1 + y\tan\phi_1),$$

so that
$$\theta_3 = \pi/2 + \theta_2 - y\sin\theta_2\cos\theta_2\tan\phi_1$$
$$= \pi/2 - \theta_1 + y\sin\theta_1\cos\theta_1\tan\phi_1.$$

Also in this case we have

$$AzM = \pi - \phi_2 = \phi_1;$$
$$AM = \pi/2 - \omega_2 = \pi/2 - \omega_3 \text{ (approximately)};$$
$$LM = \pi/2 + \theta_2 = \pi/2 - \theta_1; \quad Az = \pi/2 - v;$$

and hence
$$\cos\theta_1 = \cot\phi_1\cot\omega_3;$$
$$\cos\theta_1\tan\chi_3/y = 2\cos\theta_1\tan\omega_3\cot\phi_1 = 2\cot^2\phi_1.$$

In this case, as before, $\tan\chi_3$ must be taken positively if the crystal be positive. We thus get

$$L_1 = \cos\theta_1 \sin\phi_1 = L_2;\ M_1 = -\cos\theta_1\cos\phi_1 = -M_2;\ N_1 = \sin\theta_1 = N_2;$$

$$S_1 = \sin\theta_1 \sin\phi_1 \cos\phi_1 = -S_2;$$

$$L_3 = \sin\theta_1 \sin\phi_1 - y\sin\theta_1 \sec\phi_1 (1 - \sin^2\theta_1 \sin^2\phi_1);$$

$$M_3 = \sin\theta_1 \cos\phi_1 + y\sin^3\theta_1 \sin\phi_1;$$

$$N_3 = -\cos\theta_1 - y\sin^2\theta_1 \cos\theta_1 \tan\phi_1;$$

$$S_3 = \cos\theta_1 \sin\phi_1 \cos\phi_1 - y\cos\theta_1 (\cos 2\phi_1 - \sin^2\theta_1 \sin^2\phi_1) \\ \pm 2y\cos^2\phi_1 \sec\theta_1,$$

the upper or lower sign being taken according as the crystal is positive or negative.

These equations give

$$L_1N_2 - L_2N_1 = M_1S_2 - M_2S_1 = 0, \text{ and hence } \theta_{12} = \pi/2;$$

also $$\frac{\tan\theta_{31}}{\tanh x}$$

$$= \frac{\cos 2\theta_1 \cos\phi_1 (q^2 - \sin^2\phi_1) + y\sin\phi_1 (q^2\sin^2\theta_1 \cos 2\theta_1 + \cos^2\theta_1 \cos 2\phi_1 - \sin^2\theta_1 \sin^2\phi_1 \cos 2\theta_1 - \sin^2\theta_1 \mp 2\cos^2\phi_1)}{\sin\phi_1 (q^2 - \sin^2\phi_1) + y\cos\phi_1 \{\sin^2\theta_1 (1 + \sin^2\phi_1 - q^2) - \cos^2\theta_1 \cos 2\phi_1 \pm 2\cos^2\phi_1\}}$$

$$= \cot\phi_1 \cot 2\theta_1 + (y/\mu^2 - 1)\operatorname{cosec}^4\phi_1 [(q^2 - \cos^2\phi_1)\sin^2\theta_1 \cos 2\theta_1 \\ + (\sin^2\phi_1 \pm 2\cos^2\phi_1)(\cos^2\theta_1 \cos 2\phi_1 - \sin^2\theta_1) \\ + \cos^2\theta_1 \cos^2\phi_1 \cos 2\theta_1 \cos 2\phi_1].$$

If δ be the difference of phase between the two reflected rays, we have $\delta = \theta_{12} - \theta_{31} = \pi/2 - \theta_{31}$, so that δ is determined by the formula first obtained. With quartz the birefringence is so small that y is practically negligible, and the difference of phase is given very approximately by $\cot\delta = \tanh x \cot\phi_1 \cot 2\theta_1$. The following table compares the results of theory and experiment for the ray F incident at an angle of 45° for various values of v.

The agreement is as close as could be expected, the differences being within the limits of experimental errors. With a crystal of spar, where the birefringence is not negligible, it has been found that for $\phi_1 = 45°$ and $v = 22° 48'$ the value of δ/π is 0·184 for the ray D and 0·200 for the ray b_1. The values given by the above theory in these two cases are 0·192 and 0·200, so that the agreement is excellent.

v	δ/π (theory)	δ/π (exp.)	Difference
15° 47′	0·852	0·840	+0·012
28° 46′	0·674	0·668	+0·006
31° 15′	0·614	0·612	+0·002
44° 50′	0·286	0·284	+0·002
53° 18′	0·186	0·194	−0·008
68° 50′	0·126	0·142	−0·016

Reflection at a twin plane of a crystal.

If the media on each side of an interface are each portions of the same uniaxal crystal, and the optic axes of the two portions are in a plane at right angles to the interface and make angles α and $-\alpha$ with the interface, then the interface is called a twin plane of the crystal. The features of the reflection at such a twin plane can be discussed by means of the principles used in this chapter. In general the potential energy is given by an expression of the form

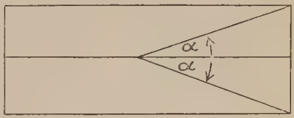

Fig. 72.

$$W = \tfrac{1}{2}\int (Af^2 + Bg^2 + Ch^2 + 2Fgh + 2Ghf + 2Hfg)\, d\tau.$$

In the present case we shall take $x=0$ as the interface, and $z=0$ as the plane containing the optic axes of the two portions of the crystal. There is then perfect symmetry with respect to the plane $z=0$, and hence we must have $F=G=0$. In the upper medium

$$W = \tfrac{1}{2}\int (Af^2 + 2Hfg + Bg^2 + Ch^2)\, d\tau,$$

and in the lower medium

$$W = \tfrac{1}{2}\int (Af^2 - 2Hfg + Bg^2 + Ch^2)\, d\tau.$$

Applying the Principle of Action to the first medium we get the dynamical equations

$$\ddot{\xi}=\frac{\partial}{\partial z}(Hf+Bg)-C\frac{\partial h}{\partial y};\qquad \ddot{\eta}=C\frac{\partial h}{\partial x}-\frac{\partial}{\partial z}(Af+Hg);$$

$$\ddot{\zeta}=\frac{\partial}{\partial y}(Af+Hg)-\frac{\partial}{\partial x}(Hf+Bg);$$

while the boundary conditions require the continuity of

$$n(Hf+Bg)-mCh,\quad lCh-n(Af+Hg),$$

and $$m(Af+Hg)-l(Hf+Bg).$$

Since, in the present case, $x=0$ is the interface, the boundary conditions are satisfied by the continuity of Ch and $Hf+Bg$. In the lower medium H must be replaced by $-H$, hence if f_1, g_1, h_1 refer to the first medium and f_2, g_2, h_2 to the second, the boundary conditions require ξ, η, ζ to be continuous, h to be continuous, and $Hf_1+Bg_1=-Hf_2+Bg_2$.

We shall consider two special cases of the general problem of reflection at the twin plane, (I) when the plane of incidence coincides with $z=0$, the plane of the optic axes, and (II) when it is at right angles to that plane.

I (a). First let us suppose that the curl (f, g, h) is perpendicular to the plane of incidence, so that $f=g=0$, and h is independent of z. The dynamical equations are

$$\ddot{\xi}=-C\frac{\partial h}{\partial y},\qquad \ddot{\eta}=C\frac{\partial h}{\partial x},\qquad \ddot{\zeta}=0,$$

so that $\ddot{h}=C\nabla^2 h$. This equation is the same in each medium, so that if we take h to be the same in each medium the dynamical equations will be satisfied, ξ, η and ζ will be continuous at the interface, and, since $f=g=0$, the other boundary conditions will be satisfied. There is thus no discontinuity at the interface and the wave passes on without any reflection or other change.

I (b). Next suppose that the curl is parallel to the plane of incidence, so that $h=0$, and, as before, everything is independent of z. We then have

$$\xi=0,\quad \eta=0;\quad f=\frac{\partial\zeta}{\partial y},\quad g=-\frac{\partial\zeta}{\partial x},\quad h=0.$$

The dynamical equations give

$$\ddot{\zeta} = A\frac{\partial^2\zeta}{\partial y^2} - 2H\frac{\partial^2\zeta}{\partial x\,\partial y} + B\frac{\partial^2\zeta}{\partial x^2}$$

in the first medium, and the same equation with the sign of H changed in the second.

For plane waves we may take

$$\zeta_1 = e^{i(pt+lx+my)}\ \text{(incident wave)} + re^{i(pt+l'x+my)}\ \text{(reflected wave)},$$

and $$\zeta_2 = se^{i(pt+l_1x+my)}\ \text{(refracted wave)}.$$

The boundary conditions require the exponential factors to be the same for all values of t and y when $x = 0$, so that p and m are the same for all the waves. The dynamical equations give

$$p^2 = Am^2 - 2Hlm + Bl^2,$$

and $$p^2 = Am^2 - 2Hl'm + Bl'^2.$$

Equating the two values of p^2 we get

$$B(l'^2 - l^2) = 2Hm(l' - l),$$

so that either $l' = l$ or $B(l' + l) = 2Hm$. The relation $l' = l$ must be rejected, as this would make the reflected ray coincide with the incident one. Hence we have $l' = -l + 2mH/B$.

In the second medium we have similarly

$$p^2 = Am^2 + 2Hl_1m + Bl_1^2,$$

and hence $$B(l^2 - l_1^2) = 2Hm(l + l_1),$$

so that either $$l_1 = -l, \text{ or } B(l - l_1) = 2Hm.$$

The relation $l_1 = -l$ is inadmissible, as this would represent a wave travelling *towards* the interface in the same direction as the incident wave. The alternative is $l_1 = l - 2mH/B$.

At the interface $x = 0$, we have

$$f_1 = \frac{\partial\zeta}{\partial y} = im(1 + r)e^{i\omega},$$

where

$$\omega = pt + my;\ f_2 = imse^{i\omega};\ g_1 = -\frac{\partial\zeta}{\partial x} = -i(l + l'r)e^{i\omega},\ g_2 = -il_1se^{i\omega},$$

so that $1 : r : s$ are the ratios of the amplitudes of the curls in the three waves. Also the boundary conditions give $1 + r = s$, and

$$Hm(1 + r) - B(l + l'r) = -Hms - Bl_1s.$$

Substituting the values of l' and l_1 found above, the second of these equations reduces to

$$(Hm - Bl) - (Hm - Bl)\, r = (Hm - Bl)\, s,$$

i.e. to $1 - r = s$. Since then $1 + r = s$ and $1 - r = s$, we must have $r = 0$, so that there is no reflection.

II. We shall next consider the case when the plane of incidence is $y = 0$, perpendicular to that of symmetry.

For the incident wave we have

$$(\xi, \eta, \zeta) = (L, M, N)\, Ie^{i(lx+nz+pt)} = (L, M, N)\, Ie^{i\omega},$$

and similarly for the other waves, where $l^2 + n^2 = p^2/v^2$, v being the velocity of the wave, and so a function of the direction cosines of the wave normal. Since

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0,$$

we have $Ll + Nn = 0$, so that $L/n = N/(-l)$, and we may represent the incident wave by

$$(\xi, \eta, \zeta) = (n, M, -l)\, Ie^{i\omega}.$$

This gives $\quad f = -\dfrac{\partial \eta}{\partial z} = -inMIe^{i\omega}; \;\; g = i(p^2/v^2)\, Ie^{i\omega}; \;\; h = ilMIe^{i\omega};$

$$(f^2 + g^2 + h^2) = (p^2/v^2)\,[M^2 + p^2/v^2]\, I^2 e^{2i\omega} = (p^2/v^2)\,[M^2 + l^2 + n^2]\, I^2 e^{2i\omega}.$$

Substituting for f, g, h in the dynamical equations we get

$$p^2 = (p^2/v^2)\, B - nH = l^2 C + n^2 A - n\,(p^2/v^2)\, H/M.$$

As the reflected waves are receding from the surface we represent them by

$$(\xi, \eta, \zeta) = (L', M', N')\, I' e^{i(-l'x+nz+pt)} = (n, M', l')\, I' e^{i\omega'},$$

where l' is positive. The dynamical equations then give

$$p^2 = (p^2/v'^2)\, B - nM'H = l'^2 C + n^2 A - n\,(p^2/v'^2)\, H/M'.$$

Putting $p^2/v'^2 = l'^2 + n^2$ and eliminating M' from the equations just obtained, we get

$$(p^2 - l'^2 B - n^2 B)\,(p^2 - l'^2 C - n^2 A) = -n^2 H^2\,(l'^2 + n^2),$$

i.e.
$$l'^4 BC - l'^2\,\{(B + C)\, p^2 - (BC + AB + H^2)\, n^2\} + p^4 - p^2 n^2\,(A + B) + n^4\,(AB + H^2) = 0.$$

We should get exactly the same equation by eliminating M from the equation

$$(p^2/v^2)B - nH = l^2C + n^2A - n(p^2/v^2)H/M$$

obtained above when considering the incident wave. Hence one of the roots of the quadratic in l'^2 is $l'^2 = l^2$, and as l' is positive, this gives $l' = l$ and so $v' = v$. As, however, the equation in l'^2 is a quadratic, there will be another possible value of l', and consequently another reflected wave. From the theorem as to the products of the roots of a quadratic we see that the other value of l' is

$$l' = +\sqrt{p^4 - p^2n^2(A+B) + n^4(AB+H^2)}/l\sqrt{BC}.$$

Thus the two reflected waves are represented respectively by

$$(\xi, \eta, \zeta) = (n, M, l)\, I' e^{i(-lx+nz+pt)},$$

and $$(\xi, \eta, \zeta) = (n, M', l')\, I'' e^{i(-l'x+nz+pt)},$$

where l' has the value just obtained.

If we deal in exactly the same way with the transmitted waves we find them represented by

$$(\xi, \eta, \zeta) = (n, -M_1, -l)\, I_1\, e^{i(lx+nz+pt)},$$

and $$(\xi, \eta, \zeta) = (n, -M_1', -l')\, I_2\, e^{i(l'x+nz+pt)}.$$

The boundary conditions, of which only four are independent, lead to these equations:—

$$I + I' + I'' = I_1 + I_2;\ I + I' + I''M'/M = -I_1 - I_2M'/M;$$

$$I - I' - I''l'/l = I_1 + I_2l'/l;\ I - I' - I''l'M'/lM = -I_1 - I_2l'M'/lM;$$

from which we obtain

$$\frac{I'}{I} = -\frac{MM'(l'^2 - l^2)}{(l'M - lM')(l'M' - lM)}, \text{ and } \frac{I''}{I} = \frac{lM(M'+M)(l'-l)}{(l'M - lM')(l'M' - lM)}.$$

These equations are quite general; but it will be convenient to deal with them in their simpler form when the doubly refracting power of the crystal is small. In such circumstances we have $C = B = A$ nearly, H is small, and v_1 and v_2 are nearly equal. We have seen in the last chapter that for a given wave velocity in any crystal there are only two possible directions of the displacement (or of its curl), and that these two are at right angles.

Hence as $v_1 = v_2$ nearly, the lines (l, M, n) and (l', M', n) are nearly at right angles, and as the first approximation we may put

$$ll' + MM' + n^2 = 0,$$

and therefore $-MM' = ll' + n^2 = l^2 + n^2$ (nearly) $= p^2/v^2$.

Also from the equations already obtained we have

$$p^2 = (l^2 + n^2) A - nHM = (l'^2 + n^2) A - nHM';$$

and thus $$(l'^2 - l^2)/(M' - M) = nH/A,$$

and $$(l' - l)/(M' - M) = nH/2Al$$

approximately. Hence, to this order of approximation, we have

$$\frac{I'}{I} = -\frac{MM'}{M - M'}\frac{nH}{Al^2};\quad \frac{I''}{I} = \frac{M(M + M')}{M - M'}\frac{nH}{2Al^2}.$$

Let us suppose now that we have a crystal, such as has been under discussion, imbedded in an isotropic medium the surface of which is parallel to the twin plane of the crystal. We shall then have *two* waves in the isotropic medium in directions $(l, 0, n)$ and $(l', 0, n)$ and these two will be incident on the twin plane. We shall consider the components of the curl perpendicular and parallel to the plane of incidence, denoting these components respectively by P and Q.

We then have

$$P = g = i(p^2/v^2) Ie^{i\omega};\ Q = \sqrt{f^2 + h^2} = i\sqrt{l^2 + n^2}\,.\,MIe^{i\omega} = i(p/v) MIe^{i\omega};$$

so that, omitting factors that are common throughout, we may put $P = (p/v) I$ and $Q = MI$. Thus for the two incident waves we have $P = (p/v)(I + J)$; $Q = MI + M'J$. In dealing with the reflected waves corresponding to J, it must be noted that J' and J'' are to be obtained from the formulae for I' and I'' by replacing I by J, and interchanging l and l', M and M'. Thus for the reflected wave we get

$$\begin{aligned} P' &= (p/v)[I' + I'' + J' + J''] \\ &= (p/v)(MI + M'J) nH/2Al^2 = (p/v) Q\,.\,nH/2Al^2, \end{aligned}$$

and $$\begin{aligned} Q' &= M[I' + J'] + M'[I'' + J''] \\ &= -[I + J] MM'nH/2Al^2 = (p/v) P\,.\,nH/2Al^2. \end{aligned}$$

It thus appears that whatever be the polarisation of the incident light, the intensity of the reflected light is proportional to that of the incident; that if $P = Q$ and there is no permanent phase

relation between the components, then the same holds for P' and Q', in other words that if the incident light is unpolarised so also is the reflected. Further we see that if $Q=0$, then $P'=0$, while if $P=0$, $Q'=0$, so that if the incident light be polarised parallel or at right angles to the plane of incidence, the reflected light is polarised in *the opposite manner*.

In the latter part of this investigation we have tacitly assumed that the crystalline stratum imbedded in the isotropic medium is thin, for we have disregarded any retardation of phase in crossing the stratum to and from the twin plane. This retardation will not polarise any light originally unpolarised, so that if the stratum is no longer thin, it will still remain true that if the incident light is unpolarised, so also is the reflected. If, however, the incident light be polarised the passage through the crystalline stratum will tend to depolarise it, and the character of the emergent beam will depend on the thickness of this stratum. Let us suppose that the incident light is polarised so that we have only one wave (I) incident on the twin plane. Then the ratio of the intensities of the two reflected waves (I' and I'') is

$$(l^2+M^2+n^2)\,I'^2 : (l'^2+M'^2+n^2)\,I''^2$$
$$=(M^2-MM')\,I'^2 : (M'^2-MM')\,I''^2 = -4MM' : (M+M')^2.$$

Now by eliminating p^2 from the equations

$$p^2=(p^2/v^2)\,B-nH=l^2C+n^2A-n\,(p^2/v^2)\,H/M$$

obtained on p. 198, we get

$$nHM^2+M\,[(A-B)\,n^2+(C-B)\,l^2]-Hn\,(l^2+n^2)=0.$$

If we disregard the difference between l and l' we may look upon this as a quadratic equation from which the two values of M are to be found. We thus get

$$MM'=-(l^2+n^2), \text{ and } M+M'=\{(A-B)\,n^2+(C-B)\,l^2\}/Hn.$$

If then ϕ be the angle of incidence so that $\cos\phi=l/\sqrt{l^2+n^2}$, the ratio of the square roots of the intensities of the two reflected waves is $2H\sin\phi : [(A-B)\sin^2\phi+(C-B)\cos^2\phi]$. This is not negligible unless ϕ be very small, so that it is only when the angle of incidence is small, and when the incident light is polarised parallel or at right angles to the plane of incidence, that the reflected beam is fully polarised.

The above theory has been applied by Lord Rayleigh to explain the peculiar internal coloured reflection exhibited by certain crystals. In some cases the reflection appears to be almost total. The coloured light is not polarised, and if one of the crystalline plates be turned round in its own plane, without altering the angle of incidence, the reflected light vanishes twice in a revolution, viz. when the plane of incidence coincides with the plane of symmetry of the crystal. "It is certainly very extraordinary and paradoxical," says Stokes, in describing the phenomenon, "that light should suffer total or all but total reflection at a transparent stratum of the very same substance, merely differing in orientation, in which the light had been travelling, and that independently of its polarisation." It appears, however, from the above investigation that the theory accounts for the facts observed by Stokes. It also predicts certain laws not previously suspected, but afterwards confirmed experimentally by Lord Rayleigh. "Theory shows," he says, "that in the act of reflection at a twin plane, *the polarisation is reversed.* If the incident light be polarised in the plane of incidence, the reflected light is polarised in the perpendicular plane, and *vice versa.* When I first obtained this result, I thought it applicable without reservation in the actual experiment, and on trial was disappointed to find that the reflected light was nearly unpolarised, even when the incident light was fully polarised, whether in the plane of incidence or in the perpendicular plane. When, however, *the angle of incidence was diminished*, the expected phenomenon was observed, provided that the original polarisation were in, or perpendicular to, the plane of incidence. If the original polarisation were oblique, the reflected light was not fully polarised, even though the angle of incidence were small."

CHAPTER VIII.

TRANSPARENT CRYSTALLINE PLATES.

THIS chapter will be devoted to a discussion of the chief features of the phenomena observed when polarised light is transmitted through a crystalline plate and subsequently passed through an 'analyser,' which damps all displacements except those confined to a particular orbit.

Let OA (Fig. 73) be a wave front incident on the crystal. The incident wave will give rise to two waves within the plate,

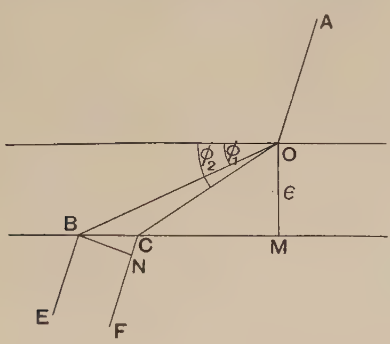

Fig. 73.

the fronts of these waves being OB and OC. These, on emergence, will produce two waves BE and CF parallel to OA. The emergent waves will thus differ in phase by an amount

$$\Delta = (2\pi/\lambda) . BN = (2\pi/\lambda) . BC \sin\phi = (2\pi/\lambda) . \sin\phi . (BM - CM)$$
$$= (2\pi\epsilon/\lambda) \sin\phi\, (\cot\phi_1 - \cot\phi_2),$$

where ϵ is the thickness of the plate, and ϕ represents as usual the angle that the wave normal makes with the normal to the interface.

If v_1 and v_2 be the velocities of the two waves in the crystal, that in air being unity, we have $\sin\phi = \sin\phi_1/v_1 = \sin\phi_2/v_2$, so that we may also express Δ in the form $\Delta = (2\pi\epsilon/\lambda)\ [\cos\phi_1/v_1 - \cos\phi_2/v_2]$, which is convenient for some purposes.

When light passes through such a crystalline medium it follows from the principles explained in Chapter VI. that the

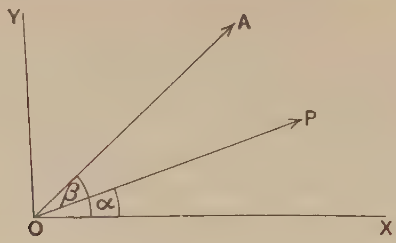

Fig. 74.

emergent waves are polarised in two directions OX and OY at right angles to one another. If the plane polariser be so placed that its principal plane makes an angle α with OX, and if A be the amplitude of the incident displacement, we may conveniently resolve this displacement into two components whose amplitudes are $A\cos\alpha$ along OX and $A\sin\alpha$ along OY. Each of these components will give rise to a separate wave within the crystal, the waves travelling with different speeds and emerging from the plate with a difference of phase Δ. There will also be a certain diminution of amplitude in the transmitted beams due to the loss of light by reflection; but in all cases to which the theory will be applied this diminution is small and is almost exactly the same for the two waves which are always very close together. Thus, as it is the relative amplitude of the two waves which mainly concerns us at present, we may with sufficient accuracy represent the emergent waves by the components $A\cos\alpha\, e^{ipt}$ and $A\sin\alpha\, e^{i(pt-\Delta)}$, or, confining ourselves to the real parts of these expressions, by $A\cos\alpha\cos pt$ and $A\sin\alpha\cos(pt-\Delta)$. On resolving the displacements along OA, the direction of the principal plane of the plane analyser, and putting $AOX = \beta$ we find that the displacement in the beam that emerges from the analyser is $A\ [\cos\alpha\cos\beta\cos pt + \sin\alpha\sin\beta\cos(pt-\Delta)]$. Thus the intensity

of the emergent beam is $I = A^2 [\cos^2(\beta - \alpha) - \sin 2\alpha \sin 2\beta \sin^2 \Delta/2]$. In a plate of uniform thickness all the quantities in this expression are constant, so that the plate appears uniformly illuminated.

The above investigation applies primarily to the case of a single incident wave; but if white light be employed we have to deal not with a single wave of definite amplitude A and wave length λ, but with an infinite number of such waves differing in amplitude and wave length. The total illumination will, in this case, be obtained from the above formula by summing the intensities due to the different waves that make up the white light. We thus find

$$\begin{aligned} I &= \cos^2(\beta - \alpha)\,\Sigma A^2 - \sin 2\alpha \,.\, \sin 2\beta \,.\, \Sigma A^2 \sin^2 \Delta/2 \\ &= I_0 - \sin 2\alpha \sin 2\beta\, \Sigma A^2 \sin^2 \Delta/2, \end{aligned}$$

where I_0 is the illumination when $\Delta = 0$, i.e. when the crystalline plate is removed and the light merely passes through the analyser.

The colours produced by transmitting light through crystalline plates were compared by Biot with those seen in Newton's rings formed by reflection from an isotropic medium. The theory of these rings has been discussed in Chapter V., and from this it appears that the intensity at normal incidence is given by the formula

$$\begin{aligned} I &= A^2 \,.\, \frac{4R^2 \sin^2 2\pi\epsilon'/\lambda}{1 + R^4 - 2R^2 \cos 4\pi\epsilon'/\lambda} = A^2 \,.\, \frac{4R^2 \sin^2 2\pi\epsilon'/\lambda}{(1 - R^2)^2 + 4R^2 \sin^2 2\pi\epsilon'/\lambda} \\ &= A^2 \,.\, \frac{(\mu^2 - 1)^2 \sin^2 2\pi\epsilon'/\lambda}{4\mu^2 + (\mu^2 - 1)^2 \sin^2 2\pi\epsilon'/\lambda}, \end{aligned}$$

where A is the amplitude of the incident wave, λ the wave length, μ the corresponding coefficient of refraction, and ϵ' the thickness of the film of air. In the case of glass for which $\mu = 1{\cdot}5$, the term $(\mu^2 - 1)^2 \sin^2 2\pi\epsilon'/\lambda$ is small compared with $4\mu^2$, so that we have approximately $I = A^2 \sin^2 2\pi\epsilon'/\lambda \,.\, (\mu^2 - 1)^2/4\mu^2$. This represents the intensity due to a single wave; but if white light were employed we should have $I = \Sigma A^2 \sin^2 2\pi\epsilon'/\lambda \,.\, (\mu^2 - 1)^2/4\mu^2$. The refractive index μ depends on the wave length, but the variation within the visible spectra is not great. If, as an approximation, we neglect the variation of μ, we have I proportional to $\Sigma A^2 \sin^2 2\pi\epsilon'/\lambda$. Biot's experiments were made at normal incidence with $\alpha = 45°$ and $\beta - \alpha = 90°$. In this case the formula found

above for the intensity after passing through a crystalline plate gives $I = \Sigma A^2 \sin^2 \Delta/2$. Hence the colours should be approximately the same as those seen in Newton's rings, provided

$$2\pi\epsilon'/\lambda = \Delta/2 = (\pi\epsilon/\lambda)(1/v_1 - 1/v_2);$$

where v_1 and v_2 as before are the velocities of the two waves in the crystal, the velocity in air being unity. This makes

$$\epsilon'/\epsilon = (1/2)(1/v_1 - 1/v_2),$$

which in many crystals is independent of the wave length to the same order of approximation as before. If then we deal with crystals that do not cause a considerable dispersion of the waves, we should expect the ratio of the thicknesses required to produce a particular colour to be nearly the same for the various colours. The following table gives the results of Biot's measurements for ten different colours.

ϵ	38·5	55·5	60·2	60·5	62·0	65·5	73·6	77·15	93·8	153·1
$\epsilon \times 9/36{\cdot}5$	9·50	13·68	14·83	14·92	15·29	16·15	18·15	19·00	23·13	37·70
ϵ'	9·70	13·55	14·67	15·10	15·10	16·25	18·17	18·70	23·20	37·50

If the above formula were exact and not merely approximate, then 9 : 36·5 would represent the constant ratio of ϵ' to ϵ and the last two rows of the table would be identical. There is thus a good agreement between theory and observation as far as the influence of the thickness of the plate on the colour is concerned.

Returning to the general formula

$$\begin{aligned} I &= \cos^2(\beta - \alpha)\,\Sigma A^2 - \sin 2\alpha \,.\, \sin 2\beta\,\Sigma A^2 \sin^2 \Delta/2 \\ &= I_0 - \sin 2\alpha \sin 2\beta\,\Sigma A^2 \sin^2 \Delta/2, \end{aligned}$$

we see that of the two terms in this expression for I the second alone depends on the lengths of the waves forming the incident beam. It is the presence of this term in the formula for I that accounts for any colour that may be observed, and so it is sometimes called the *colour term*. The colour term vanishes when $\sin 2\alpha \,.\, \sin 2\beta = 0$, which is the condition for achromatism. The lines joining points in the field of view at which this condition is satisfied are called *achromatic or neutral lines*. In crossing one of these lines the colour term changes sign, so that the quantity

in I that was formerly added to I_0 is now subtracted and *vice versa*. Hence the crossing of such a line involves a change from one colour to the complementary one. If, however, the two factors of $\sin 2\alpha \,.\, \sin 2\beta$ vanish simultaneously the neutral lines corresponding to the two factors coincide, and there is no change of sign or colour in crossing them. Such lines are called *double neutral lines.*

If we are dealing with homogeneous light there will be a family of lines in the field of view, the retardation Δ being constant along any member of the family. With most crystals the form of these lines is approximately independent of the wave length of the light. Hence if we use white light and keep to one of these lines the colour will be the same along any section of the line not crossed by a "single" neutral line. These lines are consequently spoken of as *isochromatic lines.*

A third family of lines is sometimes employed to describe the phenomena. The angle POX in Fig. 74 represents the angle (α) between the principal plane of the polariser and the direction of displacement in one of the waves (say the quicker wave) within the crystal. Points in the field of view for which this angle is the same will therefore correspond to points for which the plane of polarisation has been rotated through the same angle by the crystal. Hence the lines joining these points are called *isogyric lines.*

The last set of lines that may usefully be considered in forming a picture of the field of view are the lines of equal intensity, corresponding to $I = \text{constant}$.

In any given position the intensity is a maximum when $\Delta = 2n\pi$ and a minimum when $\Delta = (2n+1)\pi$, provided $\sin 2\alpha \,.\, \sin 2\beta$ be positive; but if $\sin 2\alpha \,.\, \sin 2\beta$ be negative the conditions for maxima and minima are interchanged. When $\beta - \alpha = 90^\circ$, I_0 vanishes and the colouring is most marked. We then have $I = \sin^2 2\alpha \,.\, \Sigma A^2 \sin^2 \Delta/2$ and there are consequently four positions of maximum intensity given by $\sin 2\alpha = \pm 1$, and four positions of zero intensity given by $\sin 2\alpha = 0$. As I vanishes with $\sin 2\alpha$ we see that, in this case, the neutral lines coincide with the lines of zero intensity. Also for any given wave length I vanishes when $\Delta = 2n\pi$ (n being any integer), so that the lines of

zero intensity are special cases of the isochromatic lines. When $\beta = \alpha$ the analyser and polariser are parallel, and

$$I = \Sigma A^2 - \sin^2 2\alpha \Sigma A^2 \sin^2 \Delta/2,$$

so that when $\sin 2\alpha = 0$ we have $I = \Sigma A^2$, representing white light. On replacing β by $90° + \beta$ in the formula for I we see that, in general, the intensity in any position of the analyser is complementary to that in the perpendicular position.

We proceed to investigate the form of some of the families of lines spoken of above in certain cases that have been examined experimentally, contenting ourselves with approximations sufficiently accurate to colligate the facts of experience.

We begin with the case of a uniaxal crystal for which, in the notation hitherto employed, we have

$$\sin \phi = \sin \phi_1/v_1 = \sin \phi_2/v_2\,; \quad v_1 = c\,; \quad v_2^2 = a^2 \sin^2 \omega_2 + c^2 \cos^2 \omega_2.$$

If the crystal be cut at right angles to the optic axis

$$\omega_1 = \phi_1 \text{ and } \omega_2 = \phi_2,$$

so that $$\sin^2 \phi_2 \operatorname{cosec}^2 \phi = v_2^2 = a^2 \sin^2 \phi_2 + c^2 \cos^2 \phi_2.$$

Hence we have $$\operatorname{cosec}^2 \phi = a^2 + c^2 \cot^2 \phi_2,$$

and $$\cot \phi_2 \sin \phi = \sqrt{1 - a^2 \sin^2 \phi}/c.$$

Similarly we should find

$$\cot \phi_1 \sin \phi = \sqrt{1 - c^2 \sin^2 \phi}/c\,;$$

and so obtain Δ in the form

$$\Delta = \sin \phi (\cot \phi_1 - \cot \phi_2)\, 2\pi\epsilon/\lambda$$
$$= (\sqrt{1 - c^2 \sin^2 \phi} - \sqrt{1 - a^2 \sin^2 \phi})\, 2\pi\epsilon/\lambda c.$$

The isochromatic curves correspond to $\Delta =$ constant, and therefore to $\phi =$ constant. If the incident light consist of a small pencil of rays proceeding from a point C (Fig. 75) at a considerable distance d from the plate, the waves near O will be approximately plane and the above investigation will be applicable. If CQ be any incident ray, O the foot of the normal from C on the plate, and $OQ = r$, then $\phi =$ constant corresponds to $r =$ constant, and the isochromatic curves are a series of concentric

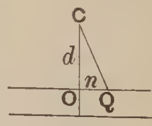

Fig. 75.

circles of which O is the centre. If the pencil is so small that $\sin^4\phi$ may be neglected, we have

$$\Delta = (a^2 - c^2)\sin^2\phi \,.\, \pi\epsilon/\lambda c = \pi\epsilon\,(a^2 - c^2)\, r^2/\lambda c d^2,$$

which expresses the relation between the radii of the rings, the thickness of the plate and the wave length of the light. The quantities a and c vary with the wave length; but this variation is slight in most cases, so that for a given retardation the radii of the rings are approximately proportional to the square root of the wave length and inversely to the square root of the thickness of the plate. When $\Delta = 2n\pi$, where n is an integer, we have r^2/λ proportional to the natural numbers, which is the same law as with Newton's rings. This special case is important because, as we have seen above, these isochromatic lines coincide with the lines of zero intensity when the polariser and analyser are at right angles. In this case, too, the lines of zero intensity also include the neutral lines given by $\sin 2\alpha = 0$, which represents two straight lines at right angles to one another, parallel and perpendicular respectively to the plane of the polariser. Fig. 76 represents the

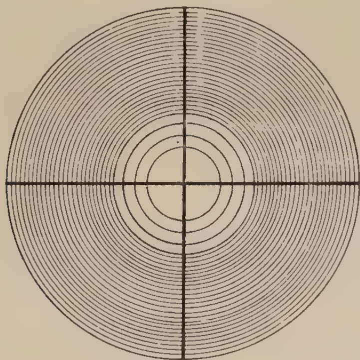

Fig. 76.

central lines of the curves of zero intensity and is drawn from a photograph of what is actually seen. It will be observed from this that there is a good agreement between theory and observation in this matter.

When the polariser and analyser are parallel it is found that the effect is complementary to that seen in the last case. This is

in accordance with the theory as stated above, as is also the fact that when $\beta - \alpha$ is neither zero nor 90°, the colour along an isochromatic circle changes into its complementary on crossing one of the neutral lines corresponding to $\alpha = 0$, $\alpha = 90°$, $\beta = 0$, $\beta = 90°$. Also when $\beta - \alpha$ is zero or 90° the neutral lines are double and there is no change of colour in crossing the lines.

Turning to the case where the crystal is cut parallel to the optic axis, we see that there are now no neutral lines. For in this case the displacements in the two waves are parallel or perpendicular to the optic axis. Thus the displacements are parallel for all points of the field, so that there could be no achromatic lines unless the polariser or analyser had their principal planes parallel or perpendicular to the optic axis. If, however, this were the case the waves transmitted through the instrument would be all of one species, i.e. either all ordinary or all extraordinary, so that there could be no interference.

To obtain the isochromatic lines we have $\cos \omega_2 = \cos v \sin \phi_2$; $\sin^2 \phi_2 \operatorname{cosec}^2 \phi = v_2^2 = a^2 - (a^2 - c^2) \cos^2 \omega_2 = a^2 - (a^2 - c^2) \cos^2 v \sin^2 \phi_2$, and therefore

$$\sin^2 \phi_2 = a^2 \sin^2 \phi / \{1 + (a^2 - c^2) \sin^2 \phi \cos^2 v\}.$$

Thus we get

$$\begin{aligned}\Delta &= \sin \phi (\cot \phi_1 - \cot \phi_2)\, 2\pi\epsilon/\lambda \\ &= [\sqrt{1 - c^2 \sin^2 \phi}/c - \sqrt{1 - \sin^2 \phi (a^2 \sin^2 v + c^2 \cos^2 v)}/a]\, 2\pi\epsilon/\lambda \\ &= [2(1/c - 1/a) + \{(a^2 \sin^2 v + c^2 \cos^2 v)/a - c\} \sin^2 \phi]\, \pi\epsilon/\lambda\end{aligned}$$

to the same order of approximation as before.

In Fig. 77 the surface of the crystal is represented by the plane

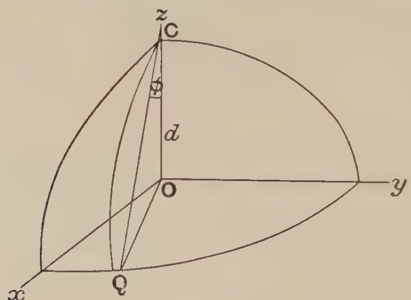

Fig. 77.

$z=0$, the axis of the crystal is along the axis of x, and CQ is any incident ray. Then COQ is the plane of incidence and xOQ the angle that the optic axis makes with this plane, an angle hitherto denoted by v. Hence if $OC=d$, as before, the coordinates of P are given by the equations $x = d \tan \phi \cos v = d \sin \phi \cos v$, and $y = d \tan \phi \sin v = d \sin \phi \sin v$, to our order of approximation. The isochromatic curves correspond to $\Delta = \text{constant}$, i.e. to $(ac - a^2 \sin^2 v - c^2 \cos^2 v) \sin^2 \phi = \text{constant}$, i.e. to $cx^2 - ay^2 = \text{constant}$. This represents a series of hyperbolas, which are very nearly rectangular, since the difference between a and c is small.

The formula for Δ may be expressed thus:

$$\Delta = \Delta_0 + \kappa (a-c)(ay^2 - cx^2),$$

where

$$\Delta_0 = (1/c - 1/a)\, 2\pi\epsilon/\lambda,$$

and

$$\kappa = \pi\epsilon/\lambda a d^2.$$

In the case of normal incidence x and y are zero and $\Delta = \Delta_0$, so that Δ_0 is the retardation of phase when the incidence is normal. By superposing one plate on another it is possible to make the retardation of phase vanish for normal incidence, and when this happens the plates are said to be compensated. When the axes of the two plates are parallel, an ordinary wave in the first remains ordinary in the second. The total difference of phase produced by the plates is

$$\begin{aligned}\Delta + \Delta' &= [\epsilon (1/c - 1/a) + \epsilon' (1/c' - 1/a')]\, 2\pi/\lambda \\ &\quad + [\epsilon (ay^2 - cx^2)(a-c)/a + \epsilon' (a'y^2 - c'x^2)(a' - c')/a']\, \pi/\lambda d^2,\end{aligned}$$

for, the plates being thin, the difference between d and d' may be neglected in the terms of the second order of small quantities. The plates will be compensated if

$$\epsilon (1/c - 1/a) = \epsilon' (1/a' - 1/c'),$$

and the isochromatic curves will be ellipses or hyperbolas according as

$$\epsilon (a-c) - \epsilon' (c' - a'),$$

and

$$\epsilon' c' (c' - a')/a' - \epsilon c (a-c)/a$$

have the same or opposite signs.

When the plates are crossed so that their axes are at right angles, the ordinary wave in the first plate becomes the extra-

ordinary wave in the second and *vice versa*. In this case v in the formulae above must be replaced by $90° + v$, so that x is replaced by $-y$ and y by x. Thus the retardation produced by the plates is

$$\Delta + \Delta' = [\epsilon(1/c - 1/a) - \epsilon'(1/c' - 1/a')]\, 2\pi/\lambda$$
$$+ [\epsilon(ay^2 - cx^2)(a-c)/a - \epsilon'(a'x^2 - c'y^2)(a'-c')/a']\, \pi/\lambda d^2.$$

If the plates be of the same material and thickness they will be compensated for normal incidence and the retardation will be

$$(a^2 - c^2)(y^2 - x^2)\, \pi\epsilon/\lambda a d^2.$$

Hence the isochromatic curves consist of a series of rectangular hyperbolas whose equations are $x^2 - y^2 = \text{constant}$. When the analyser and polariser are at right angles we have seen that the lines of zero intensity are the neutral lines given by $\sin 2\alpha = 0$, i.e. the two lines $\alpha = 0$ and $\alpha = 90°$, and the isochromatic lines $\Delta = 2n\pi$ which, as we have just seen, are the rectangular hyperbolas

$$x^2 - y^2 = \text{constant}.$$

Fig. 78 represents these dark lines drawn from a photograph and

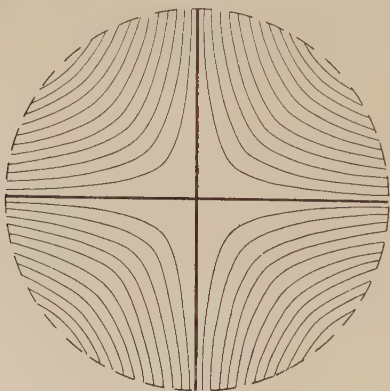

Fig. 78.

shows that there is a close agreement between theory and experiment.

So far we have been concerned with uniaxal crystals cut either at right angles or parallel to their optic axes; but the methods employed in these special cases will enable us to solve the more

general problem when the crystal is cut in any manner whatever. The position of the optic axis with reference to the interface and the plane of incidence may be determined as in the last chapter by means of the coordinates u, v or l, λ (see p. 167). We can transform from one set of coordinates to the other by means of the relations

$$\sin v = \cos\lambda \sin l,$$

and

$$\sin\lambda = \cos u \cos v,$$

while ω_2 may be expressed in terms of u and v by the formula

$$\cos\omega_2 = \cos v \cos(\phi_2 - u),$$

and in terms of l and λ by the formula

$$\cos\omega_2 = \cos\phi_2 \sin\lambda + \sin\phi_2 \cos\lambda \cos l.$$

Taking the latter formula we get

$$\begin{aligned}\sin^2\phi_2 \operatorname{cosec}^2\phi = v_2^2 &= a^2 - (a^2 - c^2)\cos^2\omega_2\\ &= a^2 - (a^2 - c^2)(\cos\phi_2 \sin\lambda + \sin\phi_2 \cos\lambda \cos l)^2,\end{aligned}$$

so that

$$\operatorname{cosec}^2\phi = a^2(1 + \cot^2\phi_2) - (a^2 - c^2)(\cot\phi_2 \sin\lambda + \cos\lambda\cos l)^2,$$

and

$$\begin{aligned}&(\sin\phi\cot\phi_2)^2[a^2\cos^2\lambda + c^2\sin^2\lambda]\\ &\quad - 2(\sin\phi\cot\phi_2)[(a^2 - c^2)\sin\lambda\cos\lambda\cos l\sin\phi]\\ &\quad - [1 - a^2\sin^2\phi + (a^2 - c^2)\cos^2\lambda\cos^2 l\sin^2\phi] = 0.\end{aligned}$$

This is a quadratic equation in $\sin\phi\cot\phi_2$, and the product of the roots is negative. Hence the equation has a positive and a negative root, of which we must take the positive one. Thus $\sin\phi\cot\phi_2$ is given definitely in terms of known quantities, and $\sin\phi\cot\phi_1$ may be found similarly, or deduced by making $a = c$. The retardation Δ is then obtained by means of the fundamental formula

$$\Delta = (\sin\phi\cot\phi_1 - \sin\phi\cot\phi_2)\,2\pi\epsilon/\lambda',$$

where λ' is the wave length, the accent being used temporarily to distinguish the wave length from the angle λ. If we wish to obtain the equation of the isochromatic curves we have to remember that l is the angle between the plane of incidence and that through the normal to the plate and the optic axis. Hence we have, to the same order of approximation as before,

$$x = d\,.\sin\phi\cos l,$$

and

$$y = d\,.\sin\phi\sin l,$$

so that the equation for $\sin\phi\cot\phi_2$ becomes

$$(\sin\phi\cot\phi_2)^2\,[a^2\cos^2\lambda + c^2\sin^2\lambda]$$
$$-\,2\sin\phi\cot\phi_2\,[\sin\lambda\cos\lambda\,.\,(a^2-c^2)\,x/d]$$
$$-\,[1 - a^2(x^2+y^2)/d^2 + \cos^2\lambda\,.\,(a^2-c^2)\,x^2/d^2] = 0.$$

The discussion of this equation is somewhat simplified by introducing the quantity v_0, the velocity of an extraordinary wave moving parallel to the face of the crystal. We have

$$v_0^2 = a^2\cos^2\lambda + c^2\sin^2\lambda,$$

so that the equation for $\sin\phi\cot\phi_2$ gives

$$v_0^2\sin\phi\cot\phi_2$$
$$= [(a^2-c^2)\sin\lambda\cos\lambda\,.\,x/d + \sqrt{v_0^2 - a^2c^2x^2/d^2 - a^2v_0^2y^2/d^2}].$$

Hence $$\Delta = (\sin\phi\cot\phi_1 - \sin\phi\cot\phi_2)\,2\pi\epsilon/\lambda'$$
$$= (A + Bx + Cx^2 + Dy^2)\,2\pi\epsilon/\lambda',$$

where

$$A = 1/c - 1/v_0\,; \qquad B = \sin 2\lambda\,.\,(c^2-a^2)/2d^2v_0^2\,;$$
$$C = (a^2c^2/v_0^3 - c)/2d^2; \quad D = (a^2/v_0 - c)/2d^2.$$

The special cases already considered are obtained from these formulae by putting $\lambda = 90°$ or 0, i.e. $v = c$ or a respectively. In general the isochromatic curves are conics whose nature depends on the coefficients B, C, and D. As the optic axis moves from the normal the angle λ decreases from 90° to zero, and the curves change from ellipses to hyperbolas. The transition takes place when $C = 0$, i.e. when

$$\sin^2\lambda = (1 - c^{2/3}/a^{2/3})/(1 - c^2/a^2),$$

in which case the curves are parabolas.

Just as in the special cases considered above we can obtain some interesting modifications of the forms of the isochromatic curves by superposing one plate upon another. If the two plates be of the same material and thickness, similarly cut, and held with their axes and principal sections parallel, the retardation is double that for a single plate. If now one plate be turned through two right angles in its own plane we must replace l by $180° + l$ when dealing with the second plate, and so replace x by $-x$. The retardation is thus

$$(A + Cx^2 + Dy^2)\,4\pi\epsilon/\lambda',$$

and the isochromatic curves are ellipses or hyperbolas with their centres at the foot of the normal from the radiant point to the plate. When the analyser and polariser are crossed a series of black spots should appear at the points of intersection of the two systems of isochromatic lines relative to each of the plates. Thus the lines will seem to be discontinuous. Fig. 79, drawn from

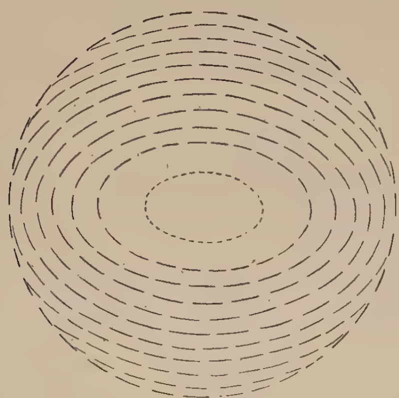

Fig. 79.

a photograph, represents the actual appearance in one case, and indicates a good agreement with the predictions of the theory.

Again, if the two plates be crossed we must replace l by $90^\circ + l$ when considering the second plate, and this will change x into $-y$ and y into x. Moreover as the waves change their species in passing from one plate to the other, the complete retardation is the difference of the retardations produced by each plate. Hence in this case we have

$$\begin{aligned}\Delta &= [(A + Bx + Cx^2 + Dy^2) - (A - By + Cy^2 + Dx^2)]\, 2\pi\epsilon/\lambda' \\ &= [B(x + y) + (C - D)(x^2 - y^2)]\, 2\pi\epsilon/\lambda'.\end{aligned}$$

The isochromatic curves are therefore hyperbolas, approximating closely to straight lines, represented by $x + y =$ constant, in the neighbourhood of the origin. These lines are a well known feature of Savart's polariscope, which consists essentially of two plates crossed in the fashion here described.

The corresponding phenomena in the case of biaxal crystals

may be dealt with in exactly the same way by aid of the formulae in Chapter VI. Thus suppose that we have a crystalline plate cut at right angles to the bisector of the angle between the optic axes, when this angle is nearly 180°. The phenomena, as might be expected, are very like those produced by a uniaxal plate cut parallel to the optic axis, and discussed on p. 210. We have seen that if v_1 and v_2 be the two wave velocities corresponding to a given direction of the wave normal, then

$$v_1^2 = b^2 + (a^2 - c^2) \cos \psi_1 \cos \psi_1' = b^2 + (a^2 - c^2) \sin \theta \sin \theta' \cos^2 \chi,$$

and $$v_2^2 = b^2 - (a^2 - c^2) \sin \theta \sin \theta' \sin^2 \chi.$$

It should be observed that the two waves in the crystal have not the same normal; but in the present case the difference of directions is so small that we may neglect it to the order of approximation with which we are being satisfied. We may then take v_1 and v_2 given by the above formulae as representing the velocities of the two waves in the crystal. Let OA_1 and OA_2 be the optic axes, ON a wave normal in the crystal, OC the normal to

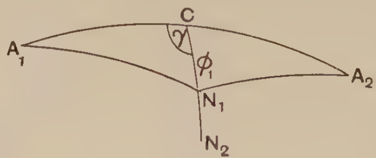

Fig. 80.

the plate. NA_1 and NA_2 are what have previously been denoted by θ and θ' respectively. Let the angle A_1CN_1 (Fig. 80) be γ,

$$A_1C = A_2C = \omega/2 = \pi/2 - \xi,$$

where ξ is a small quantity. Neglecting powers of ξ and ϕ above the second we get

$$\cos \theta = \cos \phi \cos \omega/2 + \sin \phi \cos \gamma \sin \omega/2 = \xi + \phi \cos \gamma;$$

$$\cos \theta' = \xi - \phi \cos \gamma; \quad \sin \theta \sin \theta' = 1 - \xi^2 - \phi^2 \cos^2 \gamma;$$

$$\cos \chi = 0; \quad \sin \chi = 1; \quad b^2 - c^2 = (a^2 - b^2) \xi^2;$$

$$a^2 - c^2 = (a^2 - b^2)(1 + \xi^2).$$

Hence, to this order of approximation,

$$v_1 = b, \text{ and } \sin\phi \cot\phi_1 = (1 - b^2\phi^2/2)/b,$$

$$\begin{aligned} v_2^2 &= b^2 + (a^2 - b^2)(1 + \xi^2)(1 - \xi^2 - \phi_2^2\cos^2\gamma) \\ &= b^2 + (a^2 - b^2)(1 - \phi_2^2\cos^2\gamma) \\ &= a^2 - (a^2 - b^2)\,\phi_2^2\cos^2\gamma; \end{aligned}$$

$$\sin\phi\cot\phi_2 = 1/a - \phi^2(a^2\sin^2\gamma + b^2\cos^2\gamma)/2a.$$

Making the substitutions

$$x = d\,.\tan\phi\cos\gamma = d\,.\,\phi\cos\gamma \quad\text{and}\quad y = d\,.\,\phi\sin\gamma$$

as before we get

$$\begin{aligned} \Delta &= (\sin\phi\cot\phi_1 - \sin\phi\cot\phi_2)\,2\pi\epsilon/\lambda \\ &= (a-b)\,[1/ab + (ay^2 - bx^2)/2ad^2]\,2\pi\epsilon/\lambda. \end{aligned}$$

Thus the isochromatic curves are hyperbolas, as in the case of a uniaxal plate cut parallel to the optic axis. In this case, too, there are not, as a rule, any achromatic lines. For if we take ON, the normal to the plate, as axis of z, the optic axes are nearly coincident with Ox and Ox' (Fig. 81). The directions of displacement at any point are approximately along Ox and Oy, i.e.

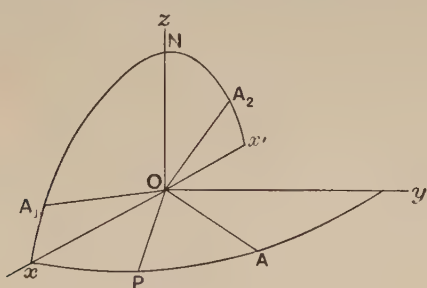

Fig. 81.

are nearly parallel for all points in the field, and these directions will not, in general, be parallel or perpendicular to the principal planes of either polariser or analyser, OP or OA.

The formulae

$$v_1^2 = b^2 + (a^2 - c^2)\cos\psi_1\cos\psi_1' = b^2 + (a^2 - c^2)\sin\theta\sin\theta'\cos^2\chi,$$

and $$v_2^2 = b^2 - (a^2 - c^2)\sin\theta\sin\theta'\sin^2\chi,$$

lead to some other results of importance. From them it appears

that whenever θ, or θ', or $a^2 - c^2$ is small, we have v_1 and v_2 nearly equal to b, and as the first approximation we may put

$$1/v_2 - 1/v_1 = (v_1 - v_2)/v_1 v_2 = (v_1^2 - v_2^2)/v_1 v_2 (v_1 + v_2)$$
$$= (v_1^2 - v_2^2)/2b^3 = (a^2 - c^2) \sin\theta \sin\theta'/2b^3.$$

When a wave is incident nearly normally on a crystal, it gives rise to two waves with different velocities v and v', and the difference of phase on emergence is

$$\Delta = (1/v - 1/v')\, 2\pi\epsilon/\lambda.$$

It must be observed, as before, that these two waves have not the same normal; but in all cases to which we apply the formulae the error in neglecting the difference of directions of the two waves within the crystal is very small. Hence, as an approximation sufficiently close for most purposes, we may put

$$\Delta = (a^2 - c^2) \sin\theta \sin\theta' (\pi\epsilon/\lambda b^3)$$

whenever either θ, or θ', or $a^2 - c^2$ is small. In such cases the isochromatic curves are given by $\sin\theta \sin\theta' =$ constant.

As an example of this we may take the case when the optic axes are inclined at a small angle and the crystal is cut at right angles to the bisector of this angle. With the same notation as before we have (Fig. 82) $NA_1 = \theta$, $NA_2 = \theta'$, and both these

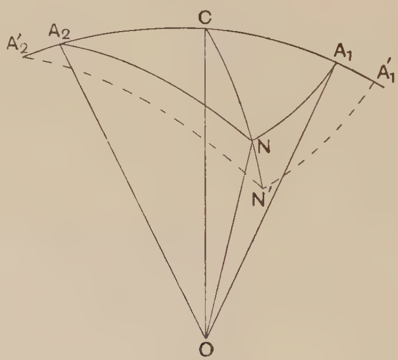

Fig. 82.

quantities are small. Thus the equation $\sin\theta \sin\theta' =$ constant, corresponds approximately to $NA_1 . NA_2 =$ constant, so that the locus of N is a lemniscate of which A_1 and A_2 are the foci.

Owing, however, to refraction the apparent positions of A_1, A_2 and N as seen from outside will be different from the real positions. These points will be shifted to A_1', A_2' and N', and as the refractive index will be very nearly constant for the different directions represented by CA_1, CA_2, and CN in the figure, the triangle $A_1'A_2'N'$ will be very nearly similar to A_1A_2N. Hence $NA_1' . NA_2'$ will be very nearly constant, and the isochromatic curves will appear as lemniscates whose foci are the apparent positions of the ends of the optic axes.

The neutral or achromatic lines are also readily obtained in this case. We have seen that in general they are determined by the condition $\sin 2\alpha . \sin 2\beta = 0$, so that an achromatic line is the locus of a point in the face of the crystal such that the displacements in the two waves in the crystal are at that point parallel or perpendicular to the principal plane of the polariser or analyser. In Chapter VI. it appears that the displacement in a wave whose normal is ON is at right angles to ON, and in a plane bisecting the angle between the planes A_1ON and A_2ON. In the present case ON is nearly at right angles to the interface, so that the displacement will be very nearly in the face of the crystal along one or other of the bisectors of the angle A_1NA_2. Let us take rectangular axes Ox and Oy (Fig. 83) parallel and perpendicular to the

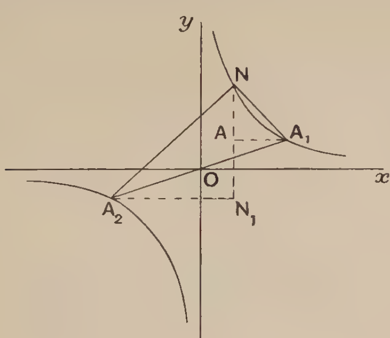

Fig. 83.

principal plane of either polariser or analyser, and the origin O at the middle point of A_1A_2, and let x_1, y_1 be the coordinates of A_1 and $-x_1, -y_1$ those of A_2. Expressing the fact that if NN_1 be

parallel to Oy the angle A_1NN_1 is equal to the angle A_2NN_1, we get

$$(x_1 - x)/(y - y_1) = (x_1 + x)/(y + y_1),$$

so that $xy = x_1y_1$. Hence the achromatic lines are rectangular hyperbolas passing through the ends of the optic axes, and having their asymptotes parallel and perpendicular to the principal plane of either analyser or polariser. If $x_1 = y_1$, so that the plane of the optic axes is inclined at an angle of 45° to the plane of the polariser or analyser, the foci of the isochromatic lines (A_1 and A_2) are the vertices of the achromatic lines. This case is represented in the left side of Fig. 84, which is drawn from a photograph and shows a close agreement between theory and observation. If either x_1 or y_1 be zero, the plane of the optic axis is parallel or perpendicular to the plane of the polariser or analyser and the achromatic lines reduce to the straight lines $xy = 0$. The right

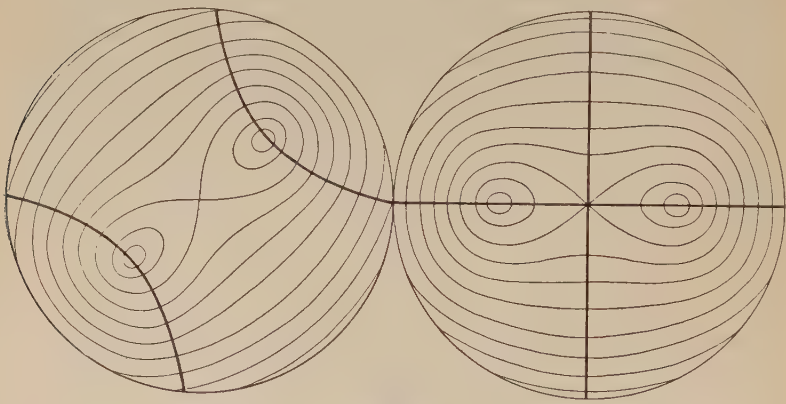

Fig. 84.

side of Fig. 84 is drawn from a photograph, showing the actual appearance of the achromatic lines in these circumstances.

It is interesting to note that if we vary the position of the analyser or polariser the achromatic and the isochromatic lines constitute 'conjugate' systems of curves*. Taking O as the origin, and the axis of x along OA_1 we have $x - c + iy = r_1e^{i\theta_1}$, and

* If $u + iv = f(x + iy)$, the curves u = constant and v = constant represent families of curves having many important relations, and styled 'conjugate.'

$x+c+iy=r_2e^{i\theta_2}$, where $OA_1=c$ and x and y are the Cartesian coordinates of any point P, (r_1, θ_1) and (r_2, θ_2) its polar coordinates referred to A_1 and A_2 as pole and the axis of x as initial line. Hence if

$$u=\log(x+iy-c)(x+iy+c)=\log r_1r_2+i(\theta_1+\theta_2),$$

we see that the curves $r_1r_2=$ constant, and $\theta_1+\theta_2=$ constant are conjugate. The first represents the isochromatic and the second the achromatic lines, for when $\theta_1+\theta_2$ is constant, the bisector of the angle A_1PA_2 is fixed in direction.

Another application of the formulae of p. 217 is met with when considering a plate cut at right angles to an optic axis OA_1. Let ON be any wave normal in the crystal near the axis OA_1, and put $\theta=A_1N=\phi_2=r$, $\theta'=A_2N$, $NA_1A_2=\psi$ (Fig. 85). We have

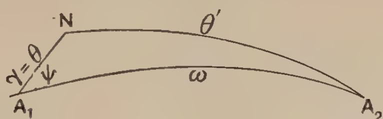

Fig. 85.

also $A_1A_2=\omega$, the angle between the optic axes, an angle given in terms of the optical constants of the crystal by the formula of p. 137. From the figure we see that

$$\cos\theta'=\cos r\cos\omega+\sin r\sin\omega\cos\psi=\cos\omega+r\sin\omega\cos\psi,$$

approximately, and hence

$$\sin\theta'=\sin\omega-r\cos\omega\cos\psi.$$

The phase difference Δ is then given by the formula

$$\Delta=(a^2-c^2)\sin\theta\,.\,\sin\theta'\,.\,\pi\epsilon/\lambda b^3$$
$$=(a^2-c^2)\left[r\sin\omega-r^2\cos\omega\cos\psi\right]\pi\epsilon/\lambda b^3,$$

and the isochromatic curves are represented by

$$r\sin\omega-r^2\cos\omega\cos\psi=\text{constant}.$$

These curves can be readily drawn from this equation, from which it is evident that the isochromatic lines are symmetrical with respect to the plane of the optic axes, and that for equal increments of Δ they are nearly equidistant circles in the immediate vicinity of A_1, but become oval shaped at greater distances from this point. The investigation on p. 219 as to the form of the

neutral lines applies also to the case now under discussion, the only difference being that the points of incidence in this case are

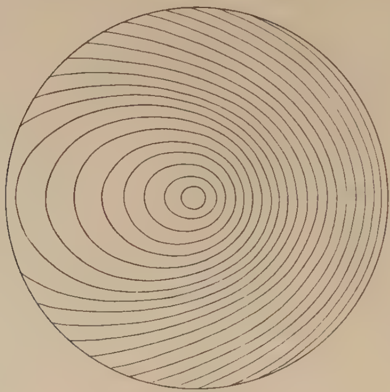

Fig. 86.

all near the end of *one* of the optic axes. A neutral line will thus consist of a portion of that branch of a rectangular hyperbola that passes through the end of the optic axis dealt with, the asymptotes of the curve being, as before, parallel and perpendicular to the principal plane of either polariser or analyser. The neutral line and the isochromatic curves for one position of the polariser and analyser are shown in Fig. 86 drawn from a photograph.

The possibility of most of the approximations that we have used in this chapter arises from the fact that the two waves, and also the two rays, within the crystal are so close to one another that they may be regarded as coincident. The results obtained with respect to the isochromatic lines may be presented in a more geometrical and somewhat more vivid manner with the aid of what is called an *Isochromatic Surface.* The exact form of presentation depends on whether we prefer to deal with waves or with rays, and the method in each case being only approximate we obtain slightly different results in the two cases.

If we are dealing with waves and take a series of wave normals OP, OP' etc. through a given point O in the crystal, then the difference of phase between the two waves that have travelled a distance ρ in the direction OP is $\Delta = \rho\,(1/v_2 - 1/v_1)$, the differ-

ence being expressed in time. The surface for which Δ is constant is an isochromatic surface. Its equation in Cartesian coordinates may be obtained by eliminating l, m, n, v_1, v_2, and ρ from the above equation for Δ, and the equations

$$x = l\rho, \quad y = m\rho, \quad z = n\rho, \quad x^2 + y^2 + z^2 = \rho^2,$$

combined with the fundamental equation

$$l^2/(v^2 - a^2) + m^2/(v^2 - b^2) + n^2/(v^2 - c^2) = 0,$$

of which v_1 and v_2 are the roots. It will be convenient, however, to obtain an approximate equation from which the general form of an isochromatic surface will be much more readily perceived than from the exact equation. In practically all crystals whose optical constants have been measured the quantity $a^2 - c^2$ is small, so that, unless great accuracy be required, we may use the approximations of p. 218, and so put

$$\Delta = \rho\,(a^2 - c^2) \sin\theta \sin\theta'/2b^3.$$

To this order of approximation the equation to an isochromatic surface takes the very simple form $\rho \sin\theta \sin\theta' = \kappa$ (a constant). This makes ρ infinite when θ or θ' is zero or π, so that the surface is asymptotic to the optic axes. In the neighbourhood of such an axis we may put $\theta' = \omega$ (the angle between the optic axes), and so get $\rho \sin\theta = \kappa \operatorname{cosec}\omega$, which represents a circular cylinder whose radius is $\kappa \operatorname{cosec}\omega$, and whose axis is along the optic axis. The distances ρ_1 and ρ_2 of the vertices of the surface from the origin are obtained by putting $\theta = \theta' = \omega/2$, and $\theta = (\pi - \omega)/2$, $\theta' = (\pi + \omega)/2$ respectively. These give

$$\rho_1 = \kappa \operatorname{cosec}^2 \omega/2, \quad \text{and} \quad \rho_2 = \kappa \sec^2 \omega/2.$$

The general form of the surface is shown in Fig. 87.

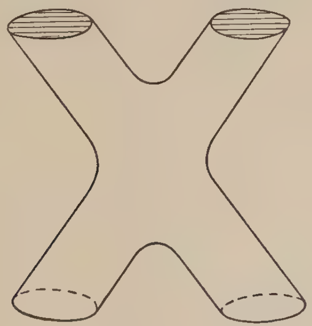

Fig. 87.

Fig. 88.

In the case of a uniaxal crystal θ and θ' coalesce, and the isochromatic surface becomes a surface of revolution, formed by turning the curve $\rho \sin^2 \theta = \kappa$ round the optic axis. The vertex is at a distance κ from the origin, and the curve approximates to a parabola for large values of ρ. The shape of the surface is indicated in Fig. 88.

We have now to see in what way the isochromatic surface enables us to find the form of the isochromatic lines in any given

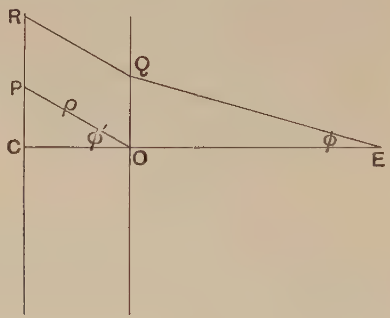

Fig. 89.

circumstances. For this purpose let a wave normal RQ (Fig. 89) in the plate be refracted so as to be received by an eye at E. Draw EOC at right angles to the plate, and OP parallel to RQ, and let ϕ and ϕ' be the angles that EQ and QR make with EOC. The difference of phase between two waves after traversing the crystal along RQ as wave normal is

$$RQ\,(1/v_2 - 1/v_1) = OP\,(1/v_2 - 1/v_1) = \rho\,(1/v_2 - 1/v_1),$$

and this, therefore, represents the difference of phase between the two waves on reaching E. If Q be any point on an isochromatic curve as seen from E, this phase difference must be constant for all points on the curve. From the figure we have

$$OQ/CP = (OE/OC)\,(\tan\phi/\tan\phi').$$

The factor OE/OC is a constant, and the ratio $\tan\phi/\tan\phi'$ is very nearly constant for small values of ϕ, even although in crystals $\sin\phi/\sin\phi'$ is not strictly constant. Hence the ratio OQ/CP is nearly constant, so that the locus of P is similar to that of Q. But P is a point of intersection of the isochromatic surface with the second surface of the crystalline plate. Hence the isochromatic

curves are similar to the sections of the isochromatic surface by the face of the crystal, except for a slight distortion when $\tan\phi/\tan\phi'$ is not constant. By taking sections of the isochromatic surface as depicted in Figs. 87 and 88 we obtain the circles, hyperbolas, and lemniscates that were obtained above in the analytical investigation of the form of the isochromatic curves in different cases.

If we prefer to deal with rays rather than with waves, we make an approximation similar to the above by supposing the two rays in the crystal to coalesce, and the light to travel along them with different *ray* velocities r_1 and r_2. The difference of phase for a ray of length ρ is then $\Delta = \rho\,(1/r_2 - 1/r_1)$, and the Cartesian equation of an isochromatic surface is obtained by eliminating L_1, M_1, N_1, r_1, r_2, and ρ from the above equation for Δ, combined with $x = L_1\rho$, $y = M_1\rho$, $z = N_1\rho$, $x^2 + y^2 + z^2 = \rho^2$, and the fact that r_1 and r_2 are the roots of the equation

$$a^2L_1^2/(r^2 - a^2) + b^2M_1^2/(r^2 - b^2) + c^2N_1^2/(r^2 - c^2) = 0.$$

This last equation gives

$$1/r_1^2 + 1/r_2^2 = \Sigma\,(1/b^2 + 1/c^2)\,L_1^2 = \Sigma\,(\mu_2^2 + \mu_3^2)\,L_1^2,$$

and
$$1/r_1^2r_2^2 = \Sigma\,L_1^2/b^2c^2 = \Sigma\,\mu_2^2\mu_3^2\,L_1^2,$$

where $\mu_1 = 1/a$, $\mu_2 = 1/b$, $\mu_3 = 1/c$, so that μ_1, μ_2, and μ_3 are the principal refractive indices. Substituting in the equation

$$(1/r_1^2 + 1/r_2^2 - \Delta^2/\rho^2)^2 = 4/r_1^2r_2^2,$$

we get

$$[(\mu_2^2 + \mu_3^2)\,x^2 + (\mu_3^2 + \mu_1^2)\,y^2 + (\mu_1^2 + \mu_2^2)\,z^2 - \Delta^2]^2$$
$$= 4\,(x^2 + y^2 + z^2)\,(\mu_2^2\mu_3^2x^2 + \mu_3^2\mu_1^2y^2 + \mu_1^2\mu_2^2z^2).$$

The isochromatic surface is therefore a surface of the fourth degree, whose form and properties can be discussed from this equation. Fig. 89 and the accompanying argument on p. 224 may be employed equally well in this case, on replacing wave normals by rays, and from this it appears that an isochromatic curve is obtained by taking a section of the isochromatic surface by the second face of the crystalline plate. The Cartesian equation just obtained lends itself very readily to this discussion; but if we wish to get an idea of the form of the surface it is convenient to use approximations similar to those on p. 223. From the formulae

of pp. 151 and 153, it appears that when $a^2 - c^2$ is small we have r_1^2 and r_2^2 each approximately equal to $1/\mu_2^2$, and

$$1/r_1^2 - 1/r_2^2 = (\mu_1^2 - \mu_3^2) \sin\phi \sin\phi',$$

where ϕ and ϕ' are the angles that the ray makes with the *ray axes*. Thus we have

$$\Delta = \rho\,(1/r_2 - 1/r_1) = \rho\,(1/r_2^2 - 1/r_1^2)/(1/r_2 + 1/r_1)$$
$$= \rho\,(\mu_3^2 - \mu_1^2) \sin\phi \sin\phi'/2\mu_2.$$

To this order of approximation the equation to an isochromatic surface takes the form $\rho \sin\phi \sin\phi' = \kappa$. This is the same as the equation on p. 223, except that the angles involved are measured from the ray axes and not from the optic axes. In the case of uniaxal crystals the ray axis and the optic axis coincide, so that the two approximations to the form of the isochromatic surface are identical. With biaxal crystals, however, the two approximations do not lead to exactly the same results, although the difference is small. The angle between the ray and the optic axes is given by the formula of p. 151. Using the values of the constants found experimentally with sodium light, this angle proves to be 29′ for mica and 53′ for aragonite, and these numbers will give some idea of the errors that may be introduced into the calculations by the use of the approximations here employed.

As another example of the errors made by disregarding the difference of directions of the two rays in the crystal, let us consider the case of nearly normal incidence on a uniaxal plate cut parallel to the axis. If ϕ be the angle of incidence we have

$$\Delta = [(\cos r_2)/v_2 - (\cos r_1)/v_1]\, 2\pi\epsilon/\lambda = [\mu_2 \cos r_2 - \mu_1 \cos r_1]\, 2\pi\epsilon/\lambda$$
$$= [(\mu_2 - \mu_1) + (1/\mu_1 - 1/\mu_2)\, \phi^2/2]\, 2\pi\epsilon/\lambda$$
$$= \Delta_0 + \phi^2 . (\mu_2 - \mu_1)\, \pi\epsilon/\lambda\mu_1\mu_2.$$

On regarding the two rays as coincident the difference of phase is

$$\Delta' = (\mu_2 - \mu_1)\, 2\pi\epsilon/\lambda \cos\phi_1 = (\mu_2 - \mu_1)(1 + \phi^2/2\mu_1^2)\, 2\pi\epsilon/\lambda$$
$$= \Delta_0 + \phi^2 (\mu_2 - \mu_1)\, \pi\epsilon/\lambda\mu_1^2.$$

Hence $(\Delta - \Delta')/(\Delta - \Delta_0) = 1 - \mu_2/\mu_1 = 0{\cdot}04$ for quartz, and $-0{\cdot}12$ for Iceland spar. This indicates the error that may be made in the order of the isochromatic fringe, and shows that such approximations cannot be safely employed when great accuracy is required. At the same time it does not alter the fact that the isochromatic

surface is most useful in enabling us to form rapidly and easily an idea of the shape of the isochromatic curves for a crystalline plate cut in any manner.

Hitherto we have supposed that the incident light is plane polarised. The phenomena are, of course, somewhat modified if circularly or elliptically polarised light be employed, but as the general principles of solution are the same in all cases it will not be necessary to do more than indicate the chief modifications to be looked for.

Taking axes OX and OY, as on p. 204, the displacements in the incident light after it has been passed through a circular polariser may be represented by $x = A\cos pt$, and $y = A\sin pt$ (as was shown in Chapter II.), where $2A^2$ is the intensity of the incident light*. The passage through the crystalline plate will produce a relative retardation Δ as before, so that on emergence we shall have displacements represented by $x = A\cos pt$, and $y = A\sin(pt - \Delta)$. Resolving along OA, the principal plane of the plane analyser, we get a displacement

$$A[\cos\beta\cos pt + \sin\beta\sin(pt - \Delta)],$$

so that the intensity of the emergent beam is

$$I = A^2(1 - \sin 2\beta \,.\, \sin\Delta).$$

If the light used be not homogeneous we have, by summation,

$$I = \Sigma A^2 - \sin 2\beta \,.\, \Sigma A^2 \sin\Delta = I_0 - \sin 2\beta \,.\, \Sigma A^2 \sin\Delta,$$

where $2I_0$ is the intensity of the incident light. A comparison of this formula with that on p. 205 will show the changes introduced by using circularly instead of plane polarised light. The main modifications are these:—

(1) The neutral or achromatic lines are given by $\sin 2\beta = 0$, instead of $\sin 2\alpha \,.\, \sin 2\beta = 0$. The absence of the lines corresponding to $\sin 2\alpha = 0$, reduces their number by one-half in the general case.

(2) In the colour term $\sin^2\Delta/2$ is replaced by $\sin\Delta$. Thus the dependence of the colour on the thickness of the plate follows a different law and the arrangement of colours is no longer like that in Newton's rings.

* This represents right-handed circular polarisation. To deal with left-handed polarisation we have merely to change the sign of p and therefore of y.

(3) As a further consequence of this change of $\sin^2 \Delta/2$ into $\sin \Delta$, there is a change in the position of the lines of maximum and minimum intensity. If $\sin 2\beta$ be positive the maximum intensity occurs when $\Delta = 2n\pi - \pi/2$, and the minimum when $\Delta = 2n\pi + \pi/2$, and if $\sin 2\beta$ be negative the positions of the maxima and minima are interchanged. Comparing this with the previous result we see that there is a difference of $\pi/2$ in the expressions for Δ in the two cases. For circularly polarised light the isochromatic lines will be intermediate in position between the corresponding lines for plane polarised light. As an illustration of this, let us take the case of a uniaxal plate cut at right angles to the optic axis. The neutral lines form a rectangular grey cross whose intensity is one-half that of the incident light, and this cross divides the plane into four quadrants as in Fig. 90. The isochromatic lines are circular, as with plane polarised light; but they are not continuous in crossing the neutral lines, being pulled in towards the centre in one pair of quadrants, and pushed out by the same amount in the other. The different appearance in the various quadrants affords, as a matter of fact, a convenient method of testing whether a crystal is positive or negative. Sin Δ is positive for crystals of one kind and negative for those of the other kind, so that the bright rings produced by one are replaced by dark rings with the other. The two views presented will be that of Fig. 90, and the same figure turned through a right angle.

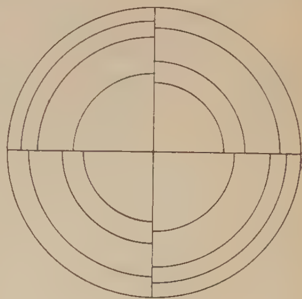

Fig. 90.

(4) As I_0 is half the intensity of the incident light, the colour term never stands alone, and there is always a wash of white present to the view.

We shall obtain still further modifications if we replace the plane analyser by a circular one, i.e. by an instrument that damps all displacements except those whose orbits are circular, right-handed, or left-handed as the case may be. After the light has passed through the circular polariser the displacement may be

represented by $x = A\cos pt$, and $y = A\sin pt$; but it is convenient to replace each of the rectilinear oscillations by a pair of circular ones thus:—

$$x = A\cos pt \text{ is equivalent to } x_1 = A/2 \,.\, \cos pt, \quad y_1 = A/2 \,.\, \sin pt,$$

and
$$x_2 = A/2 \,.\, \cos pt, \quad y_2 = -A/2 \,.\, \sin pt.$$

In the same way the displacement along OY on emergence from the plate, viz., $y = A\sin(pt - \Delta)$ is equivalent to

$$x_1' = \quad A/2 \,.\, \cos(pt - \Delta), \quad y_1' = A/2 \,.\, \sin(pt - \Delta),$$

and
$$x_2' = -A/2 \,.\, \cos(pt - \Delta), \quad y_2' = A/2 \,.\, \sin(pt - \Delta).$$

It will be observed that (x_1, y_1) and (x_1', y_1') represent right-handed, while (x_2, y_2) and (x_2', y_2') represent left-handed circular orbits. The displacement in one or other of these orbits will be damped by the analyser. Let us suppose that the second is destroyed, then the final displacement is represented by

$$x = x_1 + x_1' = A\cos\Delta/2 \,.\, \cos(pt - \Delta/2),$$

and
$$y = y_1 + y_1' = A\cos\Delta/2 \,.\, \sin(pt - \Delta/2).$$

The amplitude is therefore $A\cos\Delta/2$, and the intensity $A^2\cos^2\Delta/2$. Thus for light that is not homogeneous we have $I = \Sigma A^2\cos^2\Delta/2$, so that there are no neutral lines, and the isochromatic lines are of the same form as when a plane polariser and analyser are employed. Thus with a uniaxal crystal cut at right angles to the axis, the isochromatic lines are circles whose radii are given by the same law as on p. 209. The main differences in the appearance presented in the two cases is that with circularly analysed light there is no cross corresponding to the neutral lines, and the brightness of each ring is uniform along its circumference.

If elliptically polarised light be used the formulae are, of course, more complicated, as we then have to deal with the general case of which those previously discussed are but special examples. The usual method of producing elliptical polarisation is to pass plane polarised light through a crystalline plate, the thickness of which is chosen so that the relative retardation of the two waves that pass through the crystal is $\pi/2$, or a quarter of a wave length. Such a plate is called a quarter-wave plate. It is obvious that this arrangement gives rise to elliptical polarisation. For if the

plane of polarisation of the incident light make an angle α with the plane of least retardation of the plate, we may conveniently resolve an incident displacement whose amplitude is a into two components whose amplitudes are $a \cos \alpha$ and $a \sin \alpha$. After passing through the plate the second component has its plane changed by $\pi/2$, so that the components on emergence are represented by $\xi = a \cos \alpha \cos pt$, and $\eta = a \sin \alpha \sin pt$. On eliminating t, we get

$$\xi^2/a^2 \cos^2 \alpha + \eta^2/a^2 \sin^2 \alpha = 1,$$

so that the orbit is elliptical.

We proceed to investigate the intensity of a polarised wave, which is analysed after passing through a crystalline plate, and in order to make the resulting formula as general as possible we shall suppose that both polariser and analyser are elliptical. Let OA (Fig. 91) represent the plane of polarisation of the quicker

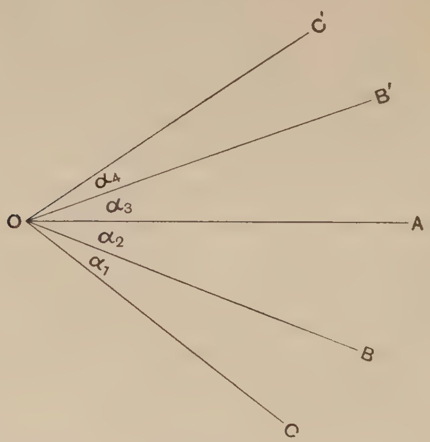

Fig. 91.

wave in the crystal, OC the plane of the plane polariser, OB that of least retardation in the quarter-wave plate that forms part of the polarising apparatus, OB' and OC' similar planes in the analyser. Let the angles between these lines be represented by α_1, α_2, α_3, and α_4 as in the figure. For brevity it will be convenient to denote $\cos \alpha$ by c, $\sin \alpha$ by s, $\cos 2\alpha$ by C, and $\sin 2\alpha$ by S. If we represent the incident wave of unit amplitude in the usual

way by e^{ipt}, then a retardation of phase Δ is represented by the introduction of the factor $e^{-i\Delta}$, and, as a special case of this, the factor $-i$ represents a retardation $\pi/2$.

Omitting the factor e^{ipt} throughout for the present, the wave that has emerged from the plane polariser may be represented by the components c_1 and s_1 along and perpendicular to OB. After passing through the quarter-wave plate, these components become c_1 and $-is_1$. Resolving these along OA and at right angles thereto, we get $c_1c_2+is_1s_2$ and $c_1s_2-is_1c_2$. After traversing the crystalline plate these become $c_1c_2+is_1s_2$ and $(c_1s_2-is_1c_2)\,e^{-i\Delta}$. Then resolving along OB' and at right angles thereto, we get

$$(c_1c_2+is_1s_2)\,c_3-(c_1s_2-is_1c_2)\,s_3e^{-i\Delta},$$

and

$$(c_1c_2+is_1s_2)\,s_3+(c_1s_2-is_1c_2)\,c_3e^{-i\Delta}$$

respectively. After passing through the quarter-wave plate of the analyser these become

$$(c_1c_2+is_1s_2)\,c_3-(c_1s_2-is_1c_2)\,s_3e^{-i\Delta},$$

and

$$-i\,(c_1c_2+is_1s_2)\,c_3-(c_1s_2-is_1c_2)\,c_3e^{-i\Delta}.$$

Finally, resolving these along OC', we get

$$c_4\left[(c_1c_2+is_1s_2)\,c_3-(c_1s_2-is_1c_2)\,s_3e^{-i\Delta}\right]$$
$$+\,s_4\left[i\,(c_1c_2+is_1s_2)\,c_3+(c_1s_2-is_1c_2)\,c_3e^{-i\Delta}\right].$$

Hence the emergent beam is represented by

$$[A+iB+(C+iD)\,e^{-i\Delta}]\,e^{ipt},$$

where

$$A=c_1c_2c_3c_4-s_1s_2s_3s_4\,;\;\; B=c_1c_3s_2s_4+c_2c_4s_1s_3\,;$$
$$C=c_2c_3s_1s_4-c_1c_4s_2s_3\,;\;\; D=c_1c_3s_2s_4+s_1s_3c_2c_4.$$

Thus the intensity I is given by

$$I=P+Q\sin\Delta+R\cos\Delta=I_0+Q\sin\Delta+N\sin^2\Delta/2,$$

where

$$2P=A^2+B^2+C^2+D^2=1+C_1C_2C_3C_4\,;$$
$$2Q=2\,(AD-BC)=S_1S_3C_4+S_2S_4C_1\,;$$
$$2R=2\,(AC+BD)=S_1S_4-C_1C_4S_2S_3\,;\;\; N=-2R\,;$$
$$I_0=P+R=(A+C)^2+(B+D)^2$$
$$=\cos^2(\alpha_1-\alpha_4)\cos^2(\alpha_2+\alpha_3)+\sin^2(\alpha_1+\alpha_4)\sin^2(\alpha_2+\alpha_3).$$

It is to be observed that the expression for I contains a term

$\sin^2 \Delta/2$ as well as $\sin \Delta$, whereas in the special cases previously considered one or other of these terms was absent.

The maximum and minimum values of the intensity are found by differentiating I with respect to Δ. This leads to

$$\tan \Delta = Q/R, \text{ and } 2I = 1 + C_1 C_2 C_3 C_4 \pm \sqrt{(1 - C_1^2 C_2^2)(1 - C_3^2 C_4^2)}.$$

All possible arrangements could be discussed by the aid of these formulae; but it will be sufficient to deal with a few simple cases that have not yet been considered. In the first place we shall suppose that the quarter-wave plates are parallel so that $\alpha_2 + \alpha_3 = 0$. We then have $2Q = S_2 \sin 2(\alpha_4 - \alpha_1)$, so that Q vanishes when $\alpha_4 = \alpha_1$ or when $\alpha_4 = \alpha_1 + 90°$. When $\alpha_4 = \alpha_1$ we get $I_0 = 1$, $N = -(1 - C_1^2 C_2^2)$, and $I = 1 - (1 - C_1^2 C_2^2) \sin^2 \Delta/2$, while when $\alpha_4 = \alpha_1 + 90°$, we get

$$I_0 = 0, \quad N = 1 - C_1^2 C_2^2, \text{ and } I = (1 - C_1^2 C_2^2) \sin^2 \Delta/2,$$

so that the intensity in this case is complementary to that in the other. Next let us suppose that the quarter-wave plates are crossed so that $\alpha_2 + \alpha_3 = 90°$. We then have

$$2Q = S_2 \sin 2(\alpha_1 + \alpha_4),$$

and Q vanishes when $\alpha_4 = -\alpha_1$ or when $\alpha_4 = -\alpha_1 + 90°$. In the first case we have $I = (1 - C_1^2 C_2^2) \sin^2 \Delta/2$, and in the second the complementary expression $I = 1 - (1 - C_1^2 C_2^2) \sin^2 \Delta/2$. In each of these four cases, then, the intensity is either $(1 - C_1^2 C_2^2) \sin^2 \Delta/2$ or its complement, so that a discussion of one case practically serves for all. If the incident light be not homogeneous, but be made up of different waves of amplitude a, we get for the intensity

$$I = \Sigma a^2 [1 - (1 - C_1^2 C_2^2) \sin^2 \Delta/2] = \Sigma a^2 - (1 - C_1^2 C_2^2) \Sigma a^2 \sin^2 \Delta/2.$$

Since $1 - C_1^2 C_2^2$ does not vanish, the colour term is never absent, and there are no achromatic lines. The isochromatic curves correspond to $\Delta =$ constant, and are bright if Δ be an even multiple of π and black if it be an odd multiple. The colour along one of the bright lines is most marked when α_2 is 45° or 135°, and least marked when α_2 is zero or 90°. The appearance presented with a uniaxal crystal cut at right angles to its axis is very similar to that described on p. 209, except that there is no cross.

We have seen that, in the general case, the minimum intensity is given by the formula

$$2I = 1 + C_1C_2C_3C_4 - \sqrt{(1-C_1^2C_2^2)(1-C_3^2C_4^2)}.$$

In order that this intensity may be zero, we must have

$$C_1C_2 + C_3C_4 = 0.$$

If θ be the angle between the quarter-wave plates, we have $\alpha_2 + \alpha_3 = \theta$, and the condition for zero intensity takes the form

$$\sin 2\alpha_2 (\sin 2\theta \cos 2\alpha_4) + \cos 2\alpha_2 (\cos 2\alpha_1 + \cos 2\theta \cos 2\alpha_4) = 0.$$

To satisfy this for all values of α_2, we must have $\sin 2\theta = 0$ and $\cos 2\alpha_1 + \cos 2\theta \cos 2\alpha_4 = 0$. The first condition requires θ to be zero or 90°, so that the quarter-wave plates must be parallel or crossed if black lines are to be seen. If $\theta = 0$, we have $\cos 2\alpha_1 + \cos 2\alpha_4 = 0$, so that $\alpha_4 = 90° - \alpha_1$ or $\alpha_4 = 90° + \alpha_1$, while if $\theta = 90°$, we have $\cos 2\alpha_1 - \cos 2\alpha_4 = 0$, so that $\alpha_4 = \alpha_1$ or $\alpha_4 = -\alpha_1$. There are thus four cases, and only four, in which black lines are possible, and it will be found that these four group themselves naturally in pairs. If $\theta = 0$ and $\alpha_4 = 90° + \alpha_1$, or if $\theta = 90°$ and $\alpha_4 = -\alpha_1$, we find $Q = 0$, and the equation $\tan \Delta = Q/R$ shows that in this case $\Delta = n\pi$. The formula for Δ in the case of a uniaxal plate cut at right angles to the axis is given on p. 208, from which it appears that $\Delta = \kappa r^2 . 2\pi/\lambda$, where κ is a quantity depending on the thickness and optical constants of the crystalline plate. Thus we have $\kappa r^2/\lambda = 2n$, so that the black lines are circles, following the same law as the curves described on p. 209, the centre (corresponding to $n = 0$) being black.

In the two other cases corresponding to $\theta = 0$, $\alpha_4 = 90° - \alpha_1$, and $\theta = 90°$, $\alpha_4 = \alpha_1$, we get $2Q = 2S_1S_2C_1$ and $2R = S_1^2 - C_1^2S_2^2$,

so that $$\frac{2 \tan \Delta/2}{1 - \tan^2 \Delta/2} = \tan \Delta = Q/R = \frac{2S_2C_1/S_1}{1 - S_2^2(C_1/S_1)^2}.$$

Solving this equation in $\tan \Delta/2$, we get

$$\tan \Delta/2 = \sin 2\alpha_2 \cot 2\alpha_1, \quad \text{or} \quad \tan \Delta/2 = -\tan 2\alpha_1 \operatorname{cosec} 2\alpha_2.$$

The first corresponds to the maxima values of I, and the second to the minima with which we are concerned at present. Thus the form of the black lines is determined from the equation $\tan \Delta/2 = -\tan 2\alpha_1 \operatorname{cosec} 2\alpha_2$. Putting $\Delta = \kappa r^2 . 2\pi/\lambda$ as before, we

get $\tan \pi\kappa r^2/\lambda = -\tan 2\alpha_1 \operatorname{cosec} 2\alpha_2$, which is the polar equation of the lines. In this equation α_1 is constant while α_2 varies with the plane of incidence of the light. Its general solution is of the form $\kappa r^2/\lambda = f(\alpha_2) + n$, where n is an integer, and from this we see that the curves approximate more and more closely to circles as their order (n) increases. The maxima and minima values of r occur when $\sin 2\alpha_2 = \pm 1$, i.e. when $\alpha_2 = \pm 45°$, and the curves are symmetrical with reference to these two lines. Their form depends on the value of the constant α_1, the accompanying figure being drawn to scale for the case $\alpha_1 = 90°$.

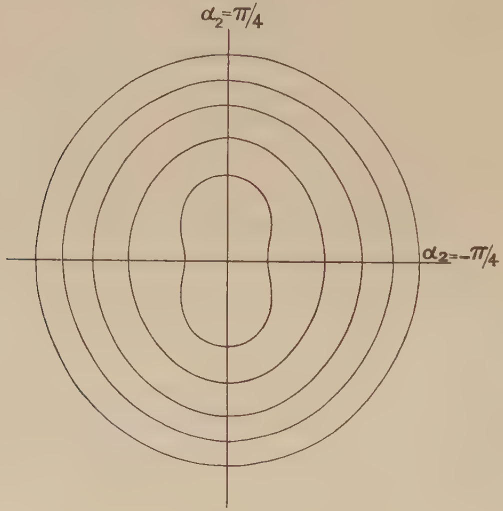

Fig. 92.

All these results as to circular and elliptical polarisation and analysation are in thorough accord with experiment, just as in the case of plane polarisation.

CHAPTER IX.

PROPAGATION OF LIGHT IN ABSORBING ISOTROPIC MEDIA. METALLIC REFLECTION.

A GENERAL method of dealing with the propagation of waves in an absorbing medium has been introduced into dynamics by Lord Rayleigh. The method consists in modifying the variation equation $\delta \int (T - W)\, dt = 0$, which expresses the Principle of Least Action, by introducing a Dissipation Function to take account of the loss of energy due to absorption. If the frictional stress between two particles of the medium be proportional to their relative velocity, and the coordinates of the particles considered be x_1, y_1, z_1, and x_2, y_2, z_2, then the components of the stress between them are

$$(\dot{x}_1 - \dot{x}_2) f_1,\ (\dot{y}_1 - \dot{y}_2) f_2,\ (\dot{z}_1 - \dot{z}_2) f_3,$$

where f_1, f_2, and f_3 are functions of the coordinates. Hence the virtual work in any small displacement of these particles is

$$(\dot{x}_1 - \dot{x}_2)\, \delta (x_1 - x_2) \,.\, f_1 + (\dot{y}_1 - \dot{y}_2)\, \delta (y_1 - y_2) \,.\, f_2 + (\dot{z}_1 - \dot{z}_2)\, \delta (z_1 - z_2) \,.\, f_3,$$

and the virtual work for the whole system is obtained from this by summation for all the particles. Now if we vary the expression

$$F = \tfrac{1}{2} \Sigma \{ (\dot{x}_1 - \dot{x}_2)^2 f_1 + (\dot{y}_1 - \dot{y}_2)^2 f_2 + (\dot{z}_1 - \dot{z}_2)^2 f_3 \}$$

with respect to the velocities alone and replace the variations of the velocities by those of the corresponding coordinates, we obtain the expression just indicated for the virtual work of the frictional stresses. Hence the Principle of Least Action leads to

$$\int (\delta T - \delta W - \delta' F)\, dt,$$

where δ' denotes that the variation is with respect to velocities only, and that, after variation, the differentials of velocities are to be replaced by those of the corresponding coordinates. Of course the Dissipation Function F may be expressed in terms of any

coordinates that may be convenient for the problem in hand, and its variation δ' found by the rule here indicated.

The only condition that limits the generality of this procedure is that the method is explicitly confined to cases where the dissipative forces are linear functions of the relative velocity of neighbouring particles; but experiment shows that this condition must be satisfied in all cases to which the theorem will be applied. If the viscous forces were not proportional to the relative velocity, the period of vibration would be a function of the amplitude, and a beam of homogeneous light passing through the medium would emerge as a mixture of different colours. This is not the case with light that passes through absorbing crystals or thin metallic plates or prisms, and as these are the only cases to which we shall have occasion to apply the method, we may do so with confidence in the basis on which it rests.

However, before any further progress can be made towards the solution of the optical problem of absorbing media, it will be necessary to specify the Dissipation Function F. And here, just as with the potential energy function W, various forms may suggest themselves, and we have to look to the results to guide us in the selection. In the present scheme the medium is of such a nature as to resist rotation only, and not distortion or compression, so that it seems natural to assume that F as well as W depends on the curl (f, g, h). In this case, in an isotropic medium, we have F of the form

$$F = \tfrac{1}{2}\int a'^2(\dot{f}^2 + \dot{g}^2 + \dot{h}^2)\,d\tau,$$

so that

$$\delta' F = \int a'^2(\dot{f}\delta f + \dot{g}\,\delta g + \dot{h}\,\delta h)\,d\tau.$$

Hence putting everything proportional to e^{ipt}, we get

$$\begin{aligned}\delta W + \delta' F &= \int\left[\left(\frac{c^2}{\mu^2} f + a'^2\dot{f}\right)\delta f + \left(\frac{c^2}{\mu^2} g + a'^2\dot{g}\right)\delta g + \left(\frac{c^2}{\mu^2} h + a'^2\dot{h}\right)\delta h\right] d\tau \\ &= \int\left(\frac{c^2}{\mu^2} + a'^2\,ip\right)(f\delta f + g\delta g + h\delta h)\,d\tau \\ &= \delta\tfrac{1}{2}\int\frac{c^2}{\mu'^2}(f^2 + g^2 + h^2)\,d\tau,\end{aligned}$$

where

$$\frac{1}{\mu'^2} = \frac{1}{\mu^2} + ip\,\frac{a'^2}{c^2}.$$

Thus the dynamical equations and boundary conditions are the same as for a transparent medium, the only difference being that μ is replaced by μ'. It will be noted that μ' is a complex of the form $\mu_0 - ia$.

This analysis is sufficient to prove that when dealing with the propagation of light in an absorbing medium we have merely to replace μ by a complex, and proceed exactly as in the case of a transparent medium; but there is one important feature not brought out by the discussion. When the theory is applied to the problem of metallic reflection, and the optical constants μ_0 and a are determined by a comparison between theory and observation, it turns out that μ^2 is such that its real part is invariably negative. This fact was a fundamental difficulty with the elastic solid theory of the ether as formerly developed, and although we have nothing to do with that theory it is none the less incumbent on us to show how the real part of μ^2 may be negative in a metal. This will necessitate the introduction of some of the terminology of electrical theory; but for present purposes little more is required than the fact that what we have called (f, g, h) is the 'displacement' of electrical theory. With this before us the relation between light and electricity appears to be of the most intimate character; the same medium and the same dynamical principles and notation are required for the elucidation of both classes of phenomena. The peculiarity of a metal, or any electrical conductor, is that it is a medium in which there are a number of free electrons whose motion constitutes the electric current. These electrons, as they move about, carry their atmosphere of strain with them, and they may be described in mathematical language as point singularities, in the neighbourhood of which the strain (f, g, h) is proportional to $(d/dx, d/dy, d/dz)(1/r)$, where r is the distance from the point x, y, z. When such point-singularities are present, the integral

$$\int (lf + mg + nh)\, dS$$

taken over any closed surface containing them will no longer be zero. If then, as on p. 37, we carry out the ideal process of smoothing out the electrons and spreading their influence uniformly throughout the medium, the vector (f, g, h) will no longer be circuital, and so cannot be regarded as the curl of the displace-

ment. However, when there are no electrons present, the rate of change of the electric 'displacement' is represented by $(\dot{f}, \dot{g}, \dot{h})$, and if we suppose that the drift of electrons added on to this 'displacement' is just sufficient to make the circuit complete (an hypothesis forced upon us by electrical considerations), we get a new vector $(\dot{f}+\dot{x}_1, \dot{g}+\dot{y}_1, \dot{h}+\dot{z}_1)$ which is circuital, x_1, y_1, z_1 being coordinates determining the position of an electron. In free ether we have seen that $W = c^2 \int (f^2+g^2+h^2)\, d\tau$, so that if we write δW in the form $\int (F\delta f + G\delta g + H\delta h)\, d\tau$, we shall have

$$(F, G, H) = c^2 (f, g, h).$$

In this notation F, G, H are generalised components of force, to be identified with the electric force of electrical theory. This force in the metallic conductor will be used up partly in accelerating the motion of the electrons, and partly in maintaining their steady drift against the resistance to migration through the medium. Thus we have $f = m\ddot{x} + \sigma\dot{x}$, where m and σ are constants. Putting all the variables proportional to e^{ipt}, we get

$$f = (mip + \sigma)\,\dot{x},$$

so that

$$\dot{x} = f/(mip+\sigma) = \dot{f}/ip\,(mip+\sigma)$$
$$= \dot{f}\,[-m/(\sigma^2+m^2p^2) - i\sigma/p\,(\sigma^2+m^2p^2)].$$

Hence $$\dot{f} + \dot{x} = \mu^2 \dot{f},$$

where $$\mu^2 = 1 - m/(\sigma^2+m^2p^2) - i\sigma/p\,(\sigma^2+m^2p^2)\text{*},$$

and similarly for the other components. In this way we obtain a vector $\mu^2(f, g, h)$ which is circuital, and which replaces the (f, g, h) of free ether. Thus the ideal process of smoothing out the electrons leads us, in this case, to a continuum in which the displacement being (ξ, η, ζ) as before, we have

$$\delta T = \int (\dot{\xi}\delta\dot{\xi} + \dot{\eta}\delta\dot{\eta} + \dot{\zeta}\delta\dot{\zeta})\, d\tau,$$

and

$$\delta W = \int (F\delta f + G\delta g + H\delta h)\, d\tau = c^2 \int (\mu^2 f\delta f + \mu^2 g\delta g + \mu^2 h\delta h)\, d\tau.$$

Since $\mu^2(f, g, h)$ is the circuital vector that replaces the (f, g, h) of free ether we have $\mu^2(f, g, h) = \text{curl}\,(\xi, \eta, \zeta)$, and therefore

* When dispersion is taken into account it may be necessary to add some other terms to μ^2, as will appear later.

$(f_1, g_1, h_1) = \text{curl}\,(\xi, \eta, \zeta)$, where $f_1 = \mu^2 f$ and so for the other components. Hence we have

$$\delta W = \frac{c^2}{\mu^2}\int (f_1 \delta f_1 + g_1 \delta g_1 + h_1 \delta h_1)\, d\tau,$$

where
$$(f_1, g_1, h_1) = \text{curl}\,(\xi, \eta, \zeta),$$

so that the dynamical equations and boundary conditions obtained from the Principle of Least Action are of exactly the same form as with a transparent medium. The only difference is that with the absorbing medium μ is complex, being given by the equation

$$\mu^2 = 1 - m/(\sigma^2 + m^2p^2) - i\sigma/p\,(\sigma^2 + m^2p^2).$$

If the second term be greater than unity, the real part of μ^2 will be negative.

Our first application of these principles will be to the interesting and important case of refraction through a metallic prism. Owing to the absorption of light by the metal, the angle of the prism must be extremely small in order that the intensity of the light transmitted may be appreciable, so that the special case of a prism of small angle (θ) is the only one that need be dealt with.

We shall suppose that the incident ray is in the plane $z = 0$, the principal plane of the prism, and when considering the first

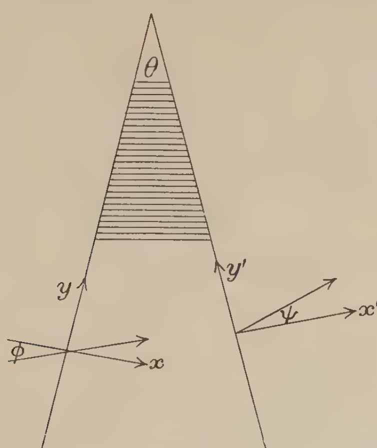

Fig. 93.

refraction we shall take the axis of x normal to the surface, and that of y along the surface towards the edge of the prism. We shall take ϕ to be the angle of incidence, and adopt the same notation as before when dealing with problems of reflection and refraction. Then, whatever be the nature of the polarisation of the incident wave, the components of the displacement in it are proportional to

$$e^{ip\,[t-(x\cos\phi+y\sin\phi)/c]},$$

and those in the refracted wave to

$$e^{ip\,[t-(nx+y\sin\phi)/c]}.$$

The dynamical equations for motion in the metal require

$$n^2+\sin^2\phi=\mu^2.$$

Before dealing with the second refraction it will be convenient to turn the axes through an angle θ, so as to measure x' and y' perpendicular and parallel respectively to the second surface. We thus have

$$x=x'\cos\theta-y'\sin\theta=x'-y'\,.\,\theta,$$

and $$y=x'\sin\theta+y'\cos\theta=x'\,.\,\theta+y',$$

since θ is small. Hence the components of the displacement within the prism are proportional to

$$e^{ip\,[t-\{n(x'-y'\,.\,\theta)+\sin\phi\,(x'\theta+y')\}/c]},\ \text{i.e. to}\ e^{ip\,[t-\{x'(n+\theta\sin\phi)+y'(\sin\phi-n\theta)\}/c]}.$$

On refraction at the second surface the components of the displacement are proportional to

$$e^{ip\,[t-(x'\cos\psi+y'\sin\psi)/c]},$$

and the condition of continuity of the displacement at all points of the surface $x'=0$ requires that $\sin\psi$ should be equal to the real part of $\sin\phi-n\theta$. Hence if n_0 be the real part of n, we have

$$\sin\psi=\sin\phi-\theta n_0.$$

Solving this by approximations we get

$$\psi=\phi-n_0\theta\sec\phi,$$

so that the deviation D is

$$D=\phi-\psi-\theta=\theta\,(n_0\sec\phi-1).$$

If, then, accurate measurements of D could be made for two different angles of incidence we should have sufficient equations

to determine both the real and the imaginary part of μ. Most of the experiments have been conducted at nearly normal incidence, for which we have $n = \mu$, so that $n_0 = \mu_0$ the real part of μ, and $D = \theta(\mu_0 - 1)$. The following table gives some of the values of μ_0 obtained by this method for different metals in sodium light.

Silver	Gold	Copper	Platinum	Iron	Nickel	Bismuth	Cobalt
0·27	0·58	0·65	1·70	1·73	2·01	2·26	2·76

The most striking feature of this table is the small value of the 'refractive index' obtained for silver, gold, and copper. In each case this is less than unity, indicating that in these metals light travels faster than in air.

In order to deal with the problem of metallic reflection we have merely to modify Fresnel's formulae given on pp. 39 and 40 by replacing μ by a complex quantity. For light polarised at right angles to the plane of incidence we have

$$r = -\tan(\phi - \phi')/\tan(\phi + \phi'),$$

and for light polarised parallel to that plane we have

$$r' = -\sin(\phi - \phi')/\sin(\phi + \phi').$$

Putting μ in the forms $\mu_0 - ia = Me^{-i\alpha}$, then μ_0 is the 'refractive index' of the medium, i.e. the ratio of the velocity of light in air to that in the medium, and a/μ_0 is the coefficient of absorption. Since

$$\sin\phi = \mu\sin\phi' = (\mu_0 - ia)\sin\phi',$$

the quantity ϕ' is complex, and we may put

$$\cos\phi' = ce^{-iu} = (1 - M^{-2}\sin^2\phi e^{2i\alpha})^{\frac{1}{2}}.$$

With this notation we get

$$r = Re^{i\theta} = \frac{\cos\phi' - \mu\cos\phi}{\cos\phi' + \mu\cos\phi} = \frac{c - M\cos\phi \,.\, e^{-i(\alpha-u)}}{c + M\cos\phi \,.\, e^{-i(\alpha-u)}}$$

$$= \frac{c - M\cos\phi\cos(\alpha - u) + iM\cos\phi\sin(\alpha - u)}{c + M\cos\phi\cos(\alpha - u) - iM\cos\phi\sin(\alpha - u)}.$$

Whence $$R^2 = \frac{1-x}{1+x}, \text{ where } x = \frac{2Mc\cos\phi\cos(\alpha-u)}{M^2\cos^2\phi + c^2},$$

and $$\tan\theta = -\frac{2Mc\cos\phi\sin(\alpha-u)}{M^2\cos^2\phi - c^2}.$$

Similarly

$$r' = R'e^{i\theta'} = \frac{\cos\phi - \mu\cos\phi'}{\cos\phi + \mu\cos\phi'} = \frac{\cos\phi - Mce^{-i(\alpha+u)}}{\cos\phi + Mce^{-i(\alpha+u)}}$$
$$= \frac{\cos\phi - Mc\cos(\alpha+u) + iMc\sin(\alpha+u)}{\cos\phi + Mc\cos(\alpha+u) - iMc\sin(\alpha+u)},$$

whence $$R'^2 = \frac{1-x'}{1+x'}, \text{ where } x' = \frac{2Mc\cos\phi\cos(\alpha+u)}{M^2c^2 + \cos^2\phi},$$

and $$\tan\theta' = -\frac{2Mc\cos\phi\sin(\alpha+u)}{M^2c^2 - \cos^2\phi}.$$

If M and α be given, these equations suffice to determine R, R', θ, and θ' completely.

For some purposes we are mainly interested in the ratio $R:R'$, and in the difference of phase $\theta-\theta'$ between the light polarised at right angles and parallel to the plane of incidence. These are readily obtained from the above formulae, for we have

$$\frac{R}{R'}e^{i(\theta-\theta')} = \frac{r}{r'} = \frac{\tan(\phi-\phi')}{\tan(\phi+\phi')}\cdot\frac{\sin(\phi+\phi')}{\sin(\phi-\phi')} = \frac{\cos(\phi+\phi')}{\cos(\phi-\phi')}$$
$$= \frac{\cos\phi\cos\phi' - \sin\phi\sin\phi'}{\cos\phi\cos\phi' + \sin\phi\sin\phi'} = \frac{Mc\cos\phi\,.\,e^{-i(\alpha+u)} - \sin^2\phi}{Mc\cos\phi\,.\,e^{-i(\alpha+u)} + \sin^2\phi}.$$

Whence $$\left(\frac{R}{R'}\right)^2 = \frac{1-y}{1+y}, \text{ where } y = \frac{2Mc\cos\phi\sin^2\phi\cos(\alpha+u)}{M^2c^2\cos^2\phi + \sin^4\phi},$$

and $$\tan(\theta'-\theta) = \frac{2Mc\cos\phi\sin^2\phi\sin(\alpha+u)}{M^2c^2\cos^2\phi - \sin^4\phi}.$$

The last equation shows that as ϕ increases from 0 to 90°, $\theta'-\theta$ increases from 0 to π; and that we have $\theta'-\theta = \pi/2$ when $M^2c^2\cos^2\phi = \sin^4\phi$; and this equation accordingly determines the *Principal Incidence*. For this angle we have

$$\left(\frac{R}{R'}\right)^2 = \frac{1-\cos(\alpha+u)}{1+\cos(\alpha+u)} = \tan^2\tfrac{1}{2}(\alpha+u) = \tan^2\beta,$$

where $\beta = (\alpha+u)/2$, and is the *Principal Azimuth*.

Before comparing these formulae with the results of experiments on metallic reflection, it will be convenient to make some transformations that will be useful for other purposes. When the optical constants M and α are determined from experiment it appears that for the metals M^2 is always large. This enables us to expand some of the above functions in ascending powers of $1/M^2$ and so to obtain approximate formulae that are sufficiently accurate for many purposes.

We have

$$ce^{-iu} = \cos\phi' = (1 - M^{-2}\sin^2\phi \,.\, e^{2i\alpha})^{\frac{1}{2}}$$
$$= 1 - \frac{\sin^2\phi \,.\, e^{2i\alpha}}{2M^2} - \frac{\sin^4\phi \,.\, e^{4i\alpha}}{8M^4} - \frac{\sin^6\phi \,.\, e^{6i\alpha}}{16M^6} - \dots.$$

If we multiply each side of this identity by $e^{i\alpha}$ and by $e^{-i\alpha}$, and equate real parts, we get

$$c\cos(\alpha - u) = \cos\alpha - \frac{\sin^2\phi}{2M^2}\cos 3\alpha - \frac{\sin^4\phi}{8M^4}\cos 5\alpha - \dots$$

$$c\cos(\alpha + u) = \cos\alpha - \frac{\sin^2\phi}{2M^2}\cos\alpha - \frac{\sin^4\phi}{8M^4}\cos 3\alpha - \dots$$

Also we have

$$c^2 = \left[1 - \frac{2\sin^2\phi\cos 2\alpha}{M^2} + \frac{\sin^4\phi}{M^4}\right]^{\frac{1}{2}}$$
$$= 1 - \frac{\sin^2\phi\cos 2\alpha}{M^2} + \frac{\sin^4\phi\sin^2 2\alpha}{2M^4} + \frac{\sin^6\phi\sin 2\alpha\sin 4\alpha}{4M^6} + \dots$$

Let us now consider how R varies as ϕ increases from 0 to 90°. For brevity we shall put $M\cos\phi = p_1$, and we then have

$$x = \frac{2p_1 c\cos(\alpha - u)}{p_1^2 + c^2}.$$

As M^2 is large we have, as the first approximation, $c = 1$ and $u = 0$, so that

$$x = 2p_1\cos\alpha/(p_1^2 + 1) = 2\cos\alpha/(p_1 + p_1^{-1}).$$

This is a maximum when $p_1 = 1$, so that R is then a minimum. Hence for light polarised at right angles to the plane of incidence the intensity of the reflected light diminishes as ϕ increases, until it reaches a minimum in the neighbourhood of $p_1 = 1$, which thus

determines what is known as the *quasi-polarising angle.* When $p_1 = 1$ we have $x = \cos\alpha$ (approximately) and

$$R = \sqrt{(1-x)/(1+x)} = \tan\alpha/2.$$

For a certain class of steel we have $M^2 = 13$ and $\alpha = 53°\ 42'$. In this case $p_1 = 1$, or $\cos\phi = M^{-1}$, gives $\phi = 73°\ 54'$ as the quasi-polarising angle, and $R = 0{\cdot}5062$ as the minimum value of the amplitude of the reflected wave. This approximation is somewhat rough as we have neglected squares and higher powers of $1/M^2$. Proceeding to a higher order we get

$$x = \frac{2p_1(\cos\alpha - \frac{1}{2}M^{-2}\sin^2\phi\cos 3\alpha - \frac{1}{8}M^{-4}\sin^4\phi\cos 5\alpha)}{p_1^2 + 1 - M^{-2}\sin^2\phi\cos 2\alpha + \frac{1}{2}M^{-4}\sin^4\phi\sin^2 2\alpha}.$$

In the small terms we may put the results of the first approximation $p_1 = 1$ and $\sin^2\phi = 1 - M^{-2}$. This gives

$$x = \frac{2p_1(\cos\alpha - \frac{1}{2}M^{-2}\cos 3\alpha + \frac{1}{2}M^{-4}\cos 3\alpha - \frac{1}{8}M^{-4}\cos 5\alpha)}{p_1^2 + 1 - M^{-2}\cos 2\alpha + M^{-4}\cos 2\alpha + \frac{1}{2}M^{-4}\sin^2 2\alpha}.$$

From this we see that R is a minimum when

$$p_1^2 = 1 - M^{-2}\cos 2\alpha + M^{-4}\cos 2\alpha + \tfrac{1}{2}M^{-4}\sin^2 2\alpha.$$

With the values of M and α given above for steel, this makes the quasi-polarising angle $\phi = 73°\,43'$.

For light polarised parallel to the plane of incidence we have

$$x' = \frac{2Mc\cos\phi\cos(\alpha + u)}{M^2c^2 + \cos^2\phi}.$$

As ϕ increases the denominator alters little, as $\cos^2\phi$ is always small compared with M^2c^2, while the numerator steadily decreases. Thus R' increases steadily and has no maximum or minimum.

We have seen that the Principal Incidence is determined by the equation $\sin^4\phi = M^2c^2\cos^2\phi$. Putting

$$c = 1 - \tfrac{1}{2}M^{-2}\sin^2\phi\cos 2\alpha + \tfrac{1}{8}M^{-4}\sin^4\phi(2\sin^2 2\alpha - \cos^2 2\alpha)$$

we get as an approximate equation to determine the Principal Incidence,

$$\sec\phi = M - \tfrac{1}{2}M^{-1}\sin^2\phi\cos 2\alpha + \tfrac{1}{8}M^{-3}\sin^4\phi(2\sin^2 2\alpha - \cos^2 2\alpha)$$
$$+ M^{-1}(1 + \tfrac{1}{2}M^{-2}\sin^2\phi\cos 2\alpha).$$

This is most simply solved by successive approximations. The first approximation gives $\sec\phi = M$, so that $p_1 = M\cos\phi = 1$. To

this order of approximation the Principal Incidence and the quasi-polarising angle are the same, so that as a rule the Principal Incidence will be very near the quasi-polarising angle. The second approximation gives $\sec\phi = M + M^{-1}(1 - \frac{1}{2}\cos 2\alpha)$, and the third

$$\sec\phi = M + M^{-1}(1 - \tfrac{1}{2}\cos 2\alpha) + M^{-3}\{\cos 2\alpha + \tfrac{1}{16}(1 - 3\cos 4\alpha)\}.$$

For most of the metals the second approximation is sufficiently accurate for comparison with the results of experiment.

From the formulae hitherto derived it will appear that the two optical constants of a metal, M and α, can be calculated if the Principal Azimuth and Principal Incidence are known from experiment. At the Principal Incidence we have $Mc = \sin^2\phi/\cos\phi$ and $\alpha + u = 2\beta$. Also $c^2 \sin 2u = M^{-2}\sin^2\phi \sin 2\alpha$. Hence

$$\sin 2u \tan^2\phi = \sin 2\alpha = \sin(4\beta - 2u),$$

so that
$$\tan 2u = \sin 4\beta/(\tan^2\phi + \cos 4\beta).$$

If the Principal Incidence (ϕ) and the Principal Azimuth (β) be known, the last equation determines the value of u at the Principal Incidence, and the equation $\alpha + u = 2\beta$ then enables us to calculate the constant α. Moreover we have

$$\cot 2u = (M^2 - \sin^2\phi\cos 2\alpha)/\sin^2\phi\sin 2\alpha,$$

so that $M^2 = \sin^2\phi\sin 4\beta\operatorname{cosec} 2u$, which determines M. Thus Conroy found for steel that the Principal Incidence was $76°\,20'$ and the Principal Azimuth was $28°\,29'$. This would make

$$\alpha = 55°\,23', \quad M^2 = 15{\cdot}67\,;$$

$$\mu_0 = M\cos\alpha = 2{\cdot}249, \text{ and } a = M\sin\alpha = 3{\cdot}257.$$

Having obtained α and M, we can calculate c and u for any value of ϕ from the formulae

$$\cot 2u = M^2\operatorname{cosec}^2\phi\operatorname{cosec} 2\alpha - \cot 2\alpha,$$

and
$$c^2 = M^{-2}\sin^2\phi\sin 2\alpha\operatorname{cosec} 2u.$$

Owing, however, to the smallness of u the latter formula is not a very good one from which to determine c^2, since the variations of $\operatorname{cosec} 2u$ are very rapid, so that a small error in u will affect c^2 considerably. This difficulty may be avoided by using the formula

$$c^4 = 1 - 2M^{-2}\sin^2\phi\cos 2\alpha + M^{-4}\sin^4\phi.$$

As a numerical example of the application of these formulae we shall calculate R and R' for various values of ϕ in the case of steel, and compare the deductions from theory with Conroy's measurements. The results are set out in the following table and illustrated graphically in Figs. 94 and 95.

ϕ	u	c	R (theory)	Difference from exp.	R' (theory)	Difference from exp.
30°	0° 26′	1·003	0·726	+0·018	0·789	+0·009
40°	0° 42′	1·005	·698	·018	·811	·009
50°	0° 59′	1·006	·656	·016	·839	·006
60°	1° 16′	1·009	·597	·013	·873	·010
70°	1° 29′	1·012	·527	·012	·911	·004
75°	1° 34′	1·012	·508	·003	·932	·005
80°	1° 37′	1·014	·540	·014	·954	·004

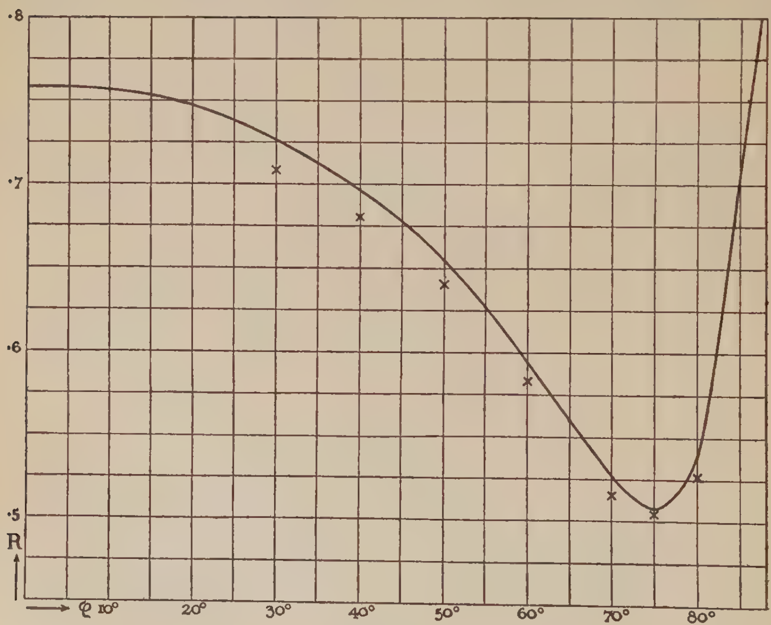

Fig. 94.

An inspection of the table or the figures will show that although the theory represents very well the general trend of the

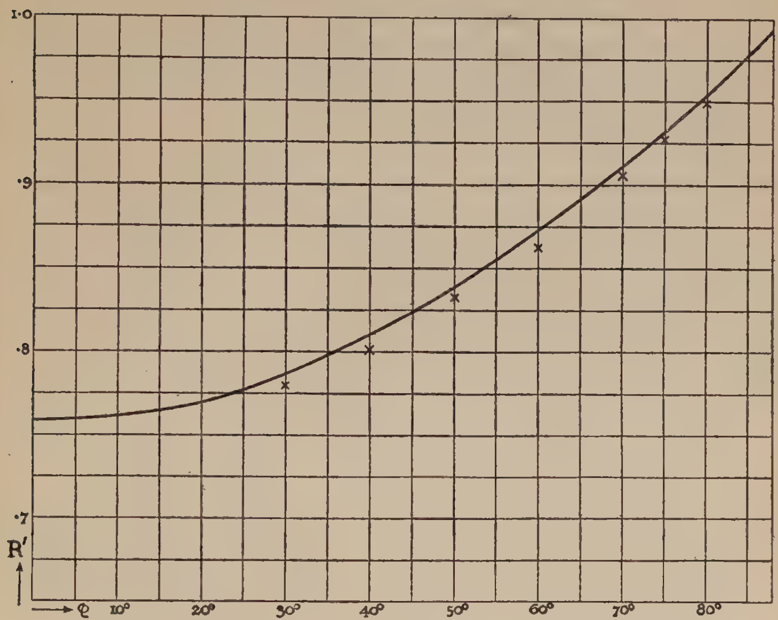

Fig. 95.

quantities R and R', the numerical agreement between theory and observation is not always very close. The values of R given by theory are almost uniformly between 1 and 2 per cent. larger than those actually found by experiment, while the values of R' are larger by quantities varying between $\frac{1}{2}$ and 1 per cent. The experimental difficulties of measuring R and R' with great accuracy are very considerable, and the differences just referred to might be ascribed to experimental errors were it not that the fact of them being all of the same sign renders this improbable, and that there are other indications that the theory of an abrupt transition when applied to the actual conditions is only an approximation to the truth.

The difference of phase between the components of the dis-

placement parallel and perpendicular to the plane of incidence is $\Delta = \theta' - \theta$, and this is given by the formula of p. 242, viz.,

$$\tan \Delta = \frac{2Mc \cos\phi \sin^2\phi \sin(\alpha + u)}{M^2c^2 \cos^2\phi - \sin^4\phi}.$$

The following table gives the values of Δ calculated from this formula and compares the results with those obtained experimentally by M. de Senarmont.

ϕ......	30°	40°	50°	60°	70°	75°	80°
Δ (theory)	·038	·069	·121	·199	·342	·462	·661
Δ (experiment)	·037	·067	·115	·178	·317	·449	·649
Difference	+ ·001	+ ·002	+ ·006	+ ·021	+ ·025	+ ·013	+ ·012

These results are exhibited graphically in Fig. 96.

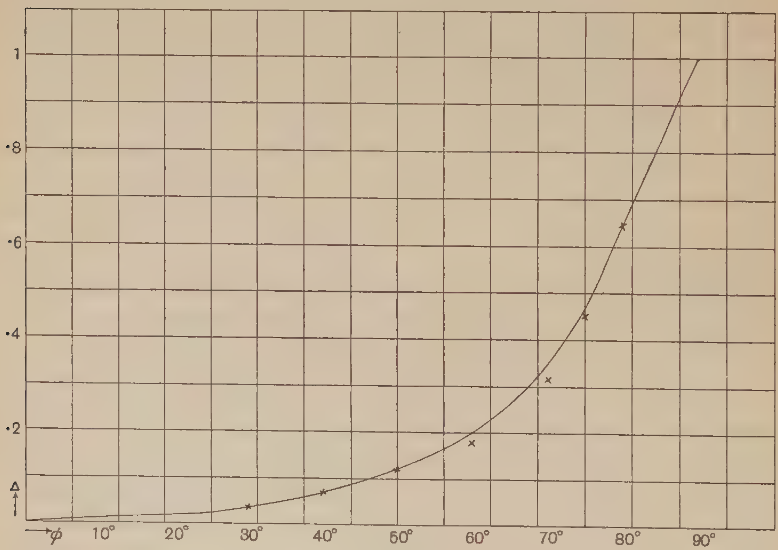

Fig. 96.

Here, again, it will be observed that the theory of an abrupt transition indicates correctly the march of Δ with the incidence, but that there is a difference between observation and theory

which is always in the same direction. Experiments with reflection from other metals, such as those of M. Meslin on gold, show still more clearly that the discrepancies in the estimates of Δ cannot be ascribed to experimental errors. In fact all careful experiments show that the character of the reflected light, as regards both amplitude and phase, depends not only on the metal employed as a reflector, but also on the condition of its surface. Hence we cannot expect to obtain a very close agreement between theory and observation unless we take into account the nature of the transition in passing into the metal—just as was found to be necessary with reflection from transparent media. Before, however, turning to the consideration of the layer of transition, we may note that the theory as developed hitherto represents the facts sufficiently closely to enable us to obtain an approximate measure of the reflecting power of a metal at normal incidence, and the change of phase produced by reflection in that case. At this incidence we have

$$R^2 = R'^2 = (1-x)/(1+x),$$

where $$x = 2M\cos\alpha/(M^2+1) = 2\mu_0/(M^2+1),$$

and $$\tan\theta = \tan\theta' = -2M\sin\alpha/(M^2-1) = -2a/(M^2-1).$$

The following table gives the values of R, R^2, and θ calculated from these formulae, the constants a and μ_0 being those found by Drude* for the reflection of sodium light from metallic surfaces that were as clean as possible. The quantity θ is expressed as a fraction of the half wave length.

	Silver	Gold	Copper	Platinum	Steel	Nickel	Bismuth	Cobalt
a	3·67	2·82	2·62	4·26	3·40	3·32	3·66	4·03
μ_0	0·18	0·37	0·64	2·06	2·41	1·79	1·90	2·12
R	·977	·923	·856	·837	·765	·787	·808	·822
R^2	·953	·851	·732	·701	·585	·620	·652	·675
$-\theta$	·169	·214	·222	·121	·125	·148	·137	·123

* Drude, *Wied. Ann.* XXXVI. p. 885, 1889; XXXIX. p. 481, 1890.

It thus appears that the reflecting power of these metals is high, which agrees with observation, and that in the case of silver the amplitude of the reflected wave is very nearly equal to that of the incident. If then we employ a silver reflector we shall produce, approximately, the stationary waves described on p. 36. The first loop will occur at a distance L_1 from the reflector, where $L_1 = -\lambda\theta/4\pi = 0{\cdot}04\lambda$, the first node at a distance $L_2 = 0{\cdot}29\lambda$, and the distance between successive loops or successive nodes will be $\lambda/2$. The difficulties of demonstrating the existence of these loops and nodes were overcome by Wiener*. He allowed a beam of light from an electric arc to fall normally on the silver coating of a plate of glass on which was placed an extremely thin film of sensitized collodion, the thickness of the film being only about one-thirtieth of a wave length. The film was very slightly inclined to the silver surface and the photograph when developed was crossed by alternately bright and dark bands, the dark bands being due to the deposit of the silver at the places where the intensity of the light was greatest. The intensity vanished at the loops of the stationary waves, indicating that the intensity as measured by the photographic effect is proportional to the mean potential energy. (See pp. 18 and 37.)

The modifications introduced by a layer of transition may be derived from the formulae of Chapter IV. by making μ a complex quantity. If we neglect squares and higher powers of the thickness of the layer, the formulae of pp. 66 and 70 give

$$r = -\frac{\mu\cos\phi - \cos\phi' + id_1(\mu^{-1}F\sin^2\phi - \mu + E\mu^2\cos\phi\cos\phi')}{\mu\cos\phi + \cos\phi' - id_1(\mu^{-1}F\sin^2\phi - \mu - E\mu^2\cos\phi\cos\phi')},$$

and

$$r' = \frac{\cos\phi - \mu\cos\phi' + id_1(\mu\cos\phi\cos\phi' - E\mu^2 + \sin^2\phi)}{\cos\phi + \mu\cos\phi' + id_1(\mu\cos\phi\cos\phi' + E\mu^2 - \sin^2\phi)}.$$

Here E and F are complex constants defined by the equations $E = \mu^{-2}\int_0^1 \mu^2 dx_1$ and $F = \mu^2\int_0^1 \mu^{-2} dx_1$. The equations for r and r' may be written in the forms $r = \dfrac{A_1 + a_1}{A_2 + a_2}$ and $r' = \dfrac{A_1' + a_1'}{A_2' + a_2'}$, where

* O. Wiener, *Wied. Ann.* XL. p. 203, 1890.

quantities such as a denote the *small* terms containing the factor d_1, and representing the corrections to the earlier formulae due to the presence of the layer of transition. We thus have

$$r = \frac{A_1}{A_2}\left[1 + \frac{2id_1 \cos\phi}{\mu^2\cos^2\phi - \cos^2\phi'}\{(E-1)\mu^2 + (F-E)\sin^2\phi\}\right],$$

and
$$r' = \frac{A_1'}{A_2'}\left[1 + \frac{2id_1(E-1)\mu^2\cos\phi}{\mu^2\cos^2\phi' - \cos^2\phi}\right].$$

To this order of approximation the corrections involve two *complex* constants, viz., $d_1(E-1)\mu^2$ and $d_1(F-E)$. As each complex requires two quantities to define it we see that the complete specification of the layer will require four constants, depending on the thickness and the law of distribution of μ^2 within the layer. However the term $(F-E)$ is associated with the factor $\sin^2\phi$, so that the product will be negligible for small angles of incidence; and even for larger values of ϕ the term $(F-E)\sin^2\phi$ may be expected to be small compared with $(E-1)\mu^2$, owing to the large value of the modulus of μ^2 for most of the metals. Hence as the first approximation we may take

$$r = \frac{A_1}{A_2}\left[1 + \frac{a\cos\phi \,.\, e^{iw}}{\mu^2\cos^2\phi - \cos^2\phi'}\right],$$

and
$$r' = \frac{A_1'}{A_2'}\left[1 + \frac{a\cos\phi \,.\, e^{iw}}{\mu^2\cos^2\phi' - \cos^2\phi}\right],$$

where $ae^{iw} = 2id_1(E-1)\mu^2$.

In the small terms we may employ the approximations $c=1$ and $u=0$, so that $\cos\phi' = 1$, and we then have

$$r = Re^{i\theta}\left[1 + \frac{a\cos\phi \,.\, e^{iw}}{\mu^2\cos^2\phi - 1}\right] = Re^{i\theta}(1 + pe^{i\lambda}),$$

and
$$r' = R'e^{i\theta'}\left[1 + \frac{a\cos\phi \,.\, e^{iw}}{\mu^2 - \cos^2\phi}\right] = R'e^{i\theta'}(1 + p'e^{i\lambda'}),$$

where R, R', θ and θ' have the same values as on p. 242. If the modulus of r be $R+\rho$ and that of r' be $R'+\rho'$, then ρ and ρ' are the corrections to the amplitudes due to the layer of transition. They are given by the formulae $\rho = Rp\cos\lambda$ and $\rho' = R'p'\cos\lambda'$. Putting $\mu\cos\phi - 1 = B_1e^{iv_1}$, and $\mu\cos\phi + 1 = B_2e^{iv_2}$, we have

$p = a \cos\phi / B_1 B_2$ and $\lambda = w - (v_1 + v_2)$. The quantities B and v are determined by the formulae

$$B_1^2 = M^2 \cos^2\phi + 1 - 2M \cos\phi \cos\alpha;$$
$$B_2^2 = M^2 \cos^2\phi + 1 + 2M \cos\phi \cos\alpha;$$
$$\tan v_1 = - M \cos\phi \sin\alpha / (M \cos\alpha \cos\phi - 1);$$
$$\tan v_2 = - M \cos\phi \sin\alpha / (M \cos\alpha \cos\phi + 1);$$

whence
$$\cos v_1 = \quad (1 - M \cos\alpha \cos\phi)/B_1;$$
$$\cos v_2 = -(1 + M \cos\alpha \cos\phi)/B_2;$$
$$\sin v_1 = M \cos\phi \sin\alpha / B_1;$$
$$\sin v_2 = M \cos\phi \sin\alpha / B_2.$$

Making these substitutions in the formulae for ρ we get

$$\rho = -\frac{Ra\cos\phi}{B_1^2 B_2^2}\left[\cos w - M^2 \cos^2\phi \,.\, \cos(2\alpha + w)\right].$$

This makes ρ vanish when $\phi = 90°$ and also when

$$\cos\phi = M^{-1}\sqrt{\cos w \sec(2\alpha + w)}.$$

In considering the law of variation of ρ, it will be convenient to ascertain the position of its maxima and minima. Putting $M \cos\phi = p_1$, we have ρ proportional to $p_1 R (p^2 - b)/B_1^2 B_2^2$, where $b = \cos w \sec(2a + w)$. Also

$$B_1^2 B_2^2 = (1 + p_1^2)^2 - 4p_1^2 \cos^2\alpha = 1 - 2p_1^2 \cos 2\alpha + p_1^4,$$

and $R = \sqrt{(1 - x)/(1 + x)}$, where $x = 2p_1 \cos\alpha/(1 + p_1^2)$. Hence we have

$$-\frac{1}{R}\frac{dR}{dp_1} = \frac{1}{1 - x^2}\frac{dx}{dp_1} = \frac{2\cos\alpha(1 - p_1^2)}{1 - 2p_1^2 \cos 2\alpha + p_1^4},$$

and the equation to determine the position of the maxima and minima of ρ is

$$\frac{3p_1^2 - b}{p_1(p_1^2 - b)} = \frac{2\cos\alpha(1 - p_1^2) + 4p_1(p_1^2 - \cos 2\alpha)}{1 - 2p_1^2 \cos 2\alpha + p_1^4},$$

i.e. $$p_1^6 - 2\cos\alpha \,.\, p_1^5 + (2\cos 2\alpha - 3b)p_1^4 + 2(1 + b)\cos\alpha \,.\, p_1^3 + (2b\cos 2\alpha - 3)p_1^2 - 2b\cos\alpha \,.\, p_1 + b = 0.$$

With the values of α and w adopted lower down for reflection from steel this equation becomes

$$p_1^6 - 1{\cdot}176p_1^5 - 0{\cdot}817p_1^4 + 1{\cdot}254p_1^3 - 3{\cdot}041p_1^2 - 0{\cdot}0783p_1 + 0{\cdot}0666 = 0.$$

Solving this, by Horner's process, we obtain two real roots, $p_1 = 0{\cdot}37$ and $p_1 = 1{\cdot}8$. The former corresponds to $\phi = 84° \; 6'$, and determines the position of the minimum, and the latter corresponds to $\phi = 60°$, and determines the position of the maximum. The march of ρ is indicated for steel in Fig. 97.

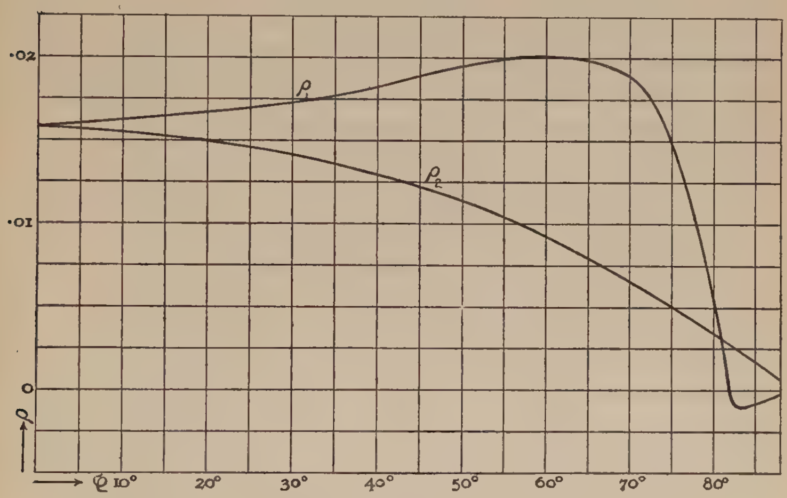

Fig. 97.

In dealing similarly with ρ' we put $\mu - \cos\phi = B_1' e^{iv_1'}$ and $\mu + \cos\phi = B_2' e^{iv_2'}$, and we then have $p' = a\cos\phi / B_1' B_2'$ and $\lambda' = w - (v_1' + v_2')$. The quantities B' and v' are determined by the formulae

$$B_1'^2 = M^2 + \cos^2\phi - 2M\cos\phi\cos\alpha;$$

$$B_2'^2 = M^2 + \cos^2\phi + 2M\cos\phi\cos\alpha;$$

$$\tan v_1' = -M\sin\alpha/(M\cos\alpha - \cos\phi);$$

$$\tan v_2' = -M\sin\alpha/(M\cos\alpha - \cos\phi);$$

where $$\rho' = -\frac{R' a \cos\phi}{B_1'^2 B_2'^2}[\cos^2\phi\cos w - M^2\cos(2\alpha + w)].$$

This makes ρ' vanish when $\phi = 90°$ and also when

$$\cos\phi = M\sqrt{\cos(2\alpha + w)\sec w},$$

but the latter equation has not, as a rule, any real root.

In the formula for ρ' the term $\cos^2\phi \cos w$ will usually be negligible in comparison with $M^2 \cos(2\alpha + w)$, so that we shall have approximately $\rho' = -R'a \cos(2\alpha + w)\cos\phi / B_1'^2 B_2'^2$. Now

$$B_1'^2 B_2'^2 = \cos^4\phi + M^4 + 2M^2\cos^2\phi\,(1 - 2\cos^2\alpha);$$

and for almost all the metals α is greater than 45°, so that $1 - 2\cos^2\alpha$ is positive. Thus $B_1'^2 B_2'^2$ is greater than M^4 and ρ' is less than $2M^{-2}R'a\cos\phi\cos(2\alpha + w)$. Thus ρ' diminishes with ϕ and is always small owing to the presence of the factor M^{-2}. If we investigate the position of its maxima and minima in the same manner as with ρ, we are led to the following equation:

$$\begin{aligned}(4M^{-6} - 3M^{-8})p_1^6 - 2M^{-6}\cos\alpha\,.\,p_1^5 + (6M^{-4}\cos 2\alpha - 4M^{-6}b + bM^{-8})p_1^4 \\ + (2M^{-2}\cos\alpha - 4M^{-4}\cos 2\alpha + 2bM^{-6}\cos\alpha)\,p_1^3 \\ - (3 - 2bM^{-4}\cos 2\alpha)\,p_1^2 - 2bM^{-2}\cos\alpha\,.\,p_1 - 6 = 0.\end{aligned}$$

Since $p_1 = M\cos\phi$ we see that p_1 cannot be greater than M, and the above equation in p_1 has no real roots less than M, so that there are no maxima and minima. The march of the function ρ' for steel is shown in Fig. 97. It will be seen from this figure that the range for which ρ and ρ' are appreciable is much larger than in the case of a transparent medium, where the influence of the transition layer is practically confined within a few degrees of the polarising angle.

For some purposes it is convenient to have formulae for the ratio of the amplitudes and the difference of phase between the components of the displacement perpendicular and parallel to the plane of incidence. These quantities ϵ and Δ are given by the formula

$$\epsilon e^{i\Delta} = \frac{r}{r'} = \frac{Re^{i\theta}(1 + pe^{i\lambda})}{R'e^{i\theta'}(1 + p'e^{i\lambda'})}.$$

Hence if
$$\frac{1 + pe^{i\lambda}}{1 + p'e^{i\lambda'}} = 1 + qe^{i\gamma} = se^{i\chi},$$

where $s = \sqrt{1 + 2q\cos\gamma + q^2}$, and $\tan\chi = q\sin\gamma/(1 + q\cos\gamma)$, we have $\epsilon = sR/R'$ and $\Delta = \theta - \theta' + \chi$. To our order of approximation we have

$$\begin{aligned}qe^{i\gamma} = pe^{i\lambda} - p'e^{i\lambda'} &= a\cos\phi\,.\,e^{iw}\left[1/(\mu^2\cos^2\phi - 1) - 1/(\mu^2 - \cos^2\phi)\right] \\ &= a\cos\phi\,.\,e^{iw}(1 + \mu^2)\sin^2\phi/(\mu^2\cos^2\phi - 1)(\mu^2 - \cos^2\phi),\end{aligned}$$

from which q and γ may be found in terms of the optical constants M, α, a and w.

Since the modulus of μ^2 is large the term $(\mu^2+1)/(\mu^2-\cos^2\phi)$ will differ very little from unity, so that as an approximation we may simplify the calculations by putting this term equal to unity in the formulae for the small corrections. In this way we get

$$qe^{i\gamma} = a\cos\phi . e^{iw}\sin^2\phi/(\mu^2\cos^2\phi - 1) = a\sin^2\phi\cos\phi . e^{iw}/B_1B_2e^{i(v_1+v_2)}.$$

Hence $$q = a\sin^2\phi\cos\phi/B_1B_2,$$

and $$\gamma = w - (v_1+v_2) = \lambda = 2\alpha + w + \gamma',$$

where $$\tan\gamma' = \sin 2\alpha/(M^2\cos^2\phi - \cos 2\alpha).$$

To determine the maximum value of q we put $M\cos\phi = p_1$ as before, and we then have to make

$$p_1(1-p_1^2/M^2)/\sqrt{1-2p_1^2\cos 2\alpha + p_1^4}$$

a maximum. This requires

$$(p_1^4 - 2p_1^2\cos 2\alpha + 1)(1-3p_1^2M^{-2}) \\ - 2p_1^2(1-p_1^2M^{-2})(p_1^2 - \cos 2\alpha) = 0.$$

On solving this by approximations, we get $p_1 = 1$ as the first approximation. This, as we have seen, is the first approximation to the quasi-polarising angle. The second approximation gives $p_1^4 = 1 - 8M^{-2}\sin^2\alpha$. Thus if $M^2 = 13$ and $\alpha = 53°\ 42'$, which are the values of the constants derived from a consideration of Conroy's experiments on reflection from steel, the first approximation gives $\phi = 73°\ 54'$ and the second $\phi = 75°\ 52'$, which is very near the Principal Incidence.

The correction to the difference of phase Δ is χ, where

$$\tan\chi = q\sin\gamma/(1+q\cos\gamma).$$

As q is small the correction is small, being greatest when q is greatest, i.e. in the neighbourhood of the Principal Incidence. It must be observed, however, that even although the correction to the change of phase be small, it may make an appreciable difference to the position of the Principal Incidence. At the Principal Incidence we have $\theta - \theta' + \chi = \pi/2$, so that $\cot(\theta-\theta') = \tan\chi$, and hence

$$\sin^4\phi - M^2c^2\cos^2\phi = 2Mc\cos\phi\sin^2\phi\sin(\alpha+u)\tan\chi.$$

This is the equation to determine the Principal Incidence, and owing to the presence of M in the term on the right-hand side, that term may be appreciable even although χ be small. This being the case, we cannot derive the optical constants M and α from observations of the Principal Azimuth and Principal Incidence alone by the formulae of p. 242. We must now proceed by successive approximations. The true values of M and α will be smaller than those obtained by neglecting the layer of transition, but the method of p. 242 will yield results that will serve as a good first approximation. We can then calculate the other constants a and w from observation of the Principal Azimuth (β) and the Principal Incidence (ϕ).

We have

$$\sqrt{1+2q\cos\gamma+q^2} = \frac{R+\rho}{R'+\rho'}\Big/\frac{R}{R'} = \frac{R'}{R}\tan\beta,$$

and
$$\frac{q\sin\gamma}{1+q\cos\gamma} = \tan\chi = \frac{\sin^4\phi - M^2c^2\cos^2\phi}{2Mc\cos\phi\sin^2\phi\sin(\alpha+u)}.$$

The second of these equations determines χ, and we then have

$$\tan\gamma = \frac{R'/R\tan\beta\sin\chi}{R'/R\tan\beta\cos\chi-1},$$

and
$$q = R'/R\,.\tan\beta\sin\chi\operatorname{cosec}\gamma.$$

These equations give us γ and q, and then the constants a and w are obtained from the formulae $q=a\sin^2\phi\cos\phi/B_1B_2$, and $\gamma=2\alpha+w+\gamma'$, where $\tan\gamma'=\sin 2\alpha/(M^2\cos^2\phi-\cos 2\alpha)$. Having obtained approximate values for a and w, we may use them as a basis for finding a closer approximation to the values of M and α, and continue this process as far as may be thought desirable. On applying this method to the consideration of Conroy's experiments on reflection from steel we find that the quantities μ_0 and a_0 (where $\mu_0 - ia_0 = \mu = Me^{-i\alpha}$) are 2·134 and 2·906 respectively, instead of 2·249 and 3·257 as given by the theory of an abrupt transition; while from Meslin's experiments on gold we get similarly 0·9 and 2·47 instead of 0·135 and 3·31. Thus the neglect of the layer of transition may introduce very considerable errors in the estimate of the optical constants of a metal.

The values of ρ and ρ' given by the method just indicated are

set out in the following table, and exhibited graphically in Fig. 97 for the case of reflection from steel dealt with experimentally by Conroy.

ϕ	ρ	ρ'	ϕ	ρ	ρ'
0°	0·016	0·016	60°	0·021	0·009
30°	·017	·014	70°	·019	·007
40°	·018	·013	75°	·015	·005
50°	·019	·011	80°	·006	·004

With the aid of these numbers we get the following table, giving the amplitudes $R+\rho$ and $R'+\rho'$ and comparing them with Conroy's measurements.

ϕ	$R+\rho$ (theory)	$R+\rho$ (exp.)	Difference	$R'+\rho'$ (theory)	$R'+\rho'$ (exp.)	Difference
30°	0·711	0·708	+0·003	0·777	0·779	−0·002
40°	·682	·680	+0·002	·800	·801	−0·001
50°	·640	·640	0	·830	·833	−0·003
60°	·582	·584	−0·002	·865	·863	+0·002
70°	·517	·515	+0·002	·906	·907	−0·001
75°	·504	·505	−0·001	·928	·928	0
80°	·543	·525	+0·018	·951	·950	+0·001

These results are represented in Figs. 98 and 99.

It will be seen that there is a very close agreement between theory and experiment, the only appreciable difference being for $R+\rho$ at an incidence of 80°, and there is good reason to doubt the accuracy of the experiments in this case.

The difference of phase between the components of the displacement that are parallel and perpendicular to the plane of incidence is $\Delta = \theta' - \theta - \chi$, where χ is determined by the formulae above. Applying these formulae to the experiments of M. de

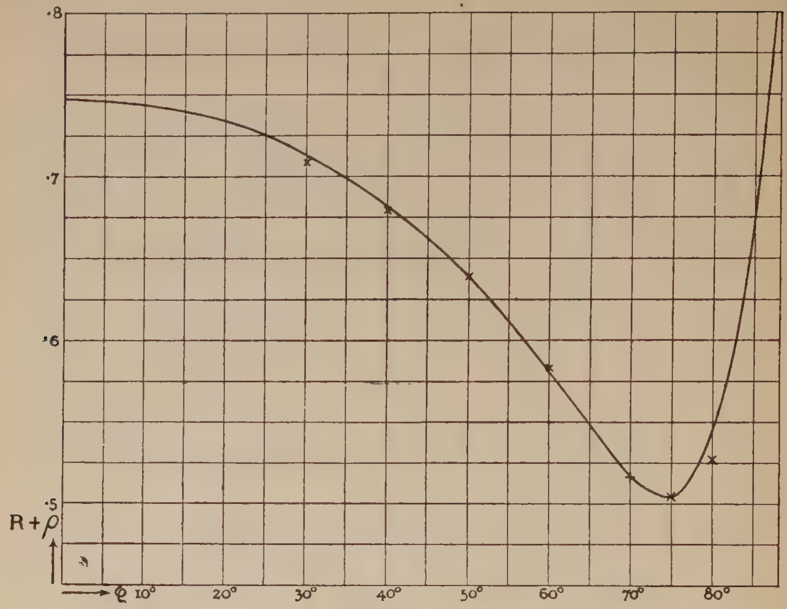

Fig. 98.

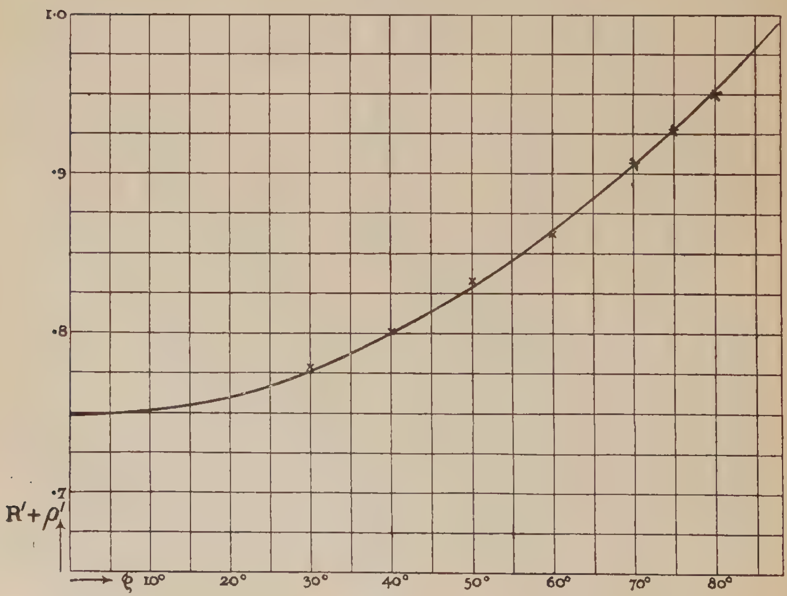

Fig. 99.

Senarmont, referred to on p. 248, we obtain the following results, which are set out in Fig. 100.

ϕ......	30°	40°	50°	60°	70°	75°	80°
Δ (theory)	·037	·068	·117	·193	·331	·447	·646
Δ (exp.)	·037	·067	·115	·178	·317	·449	·649
Difference	0	+·001	+·002	+·015	+·014	−·002	−·003

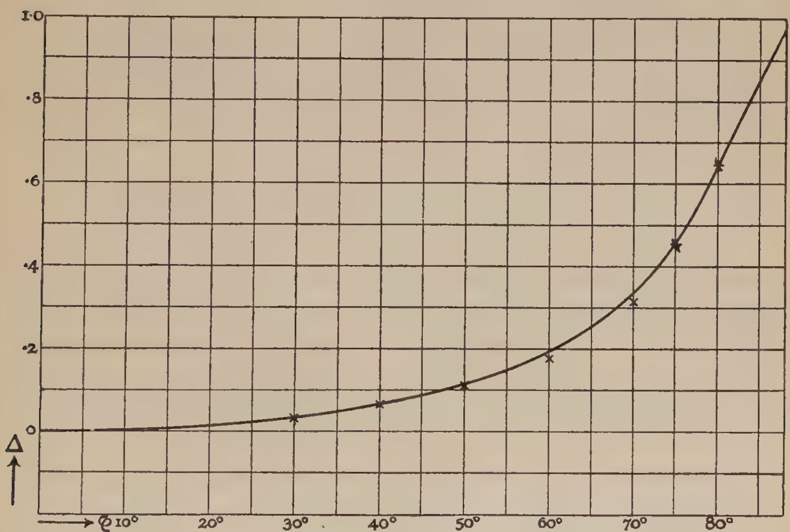

Fig. 100.

The method of determining the constants M, α, a, and w, explained on p. 255, supposes that we have before us the results of a series of experiments giving the Principal Incidence and Azimuth and the values of the amplitudes of the displacements in the reflected waves for various incidences. We may also obtain these constants from observations of the difference of phase Δ, at various angles of incidence. Theoretically we could obtain the four constants from any four observations of Δ; but the equations are much too complicated to be solved as a set of four simultaneous equations. Practically we must proceed by a series of approxima-

tions and in this way consistent results are soon obtained, unless very great accuracy be aimed at. From the formula it appears that χ (the correction to the difference of phase due to the layer) vanishes at normal incidence and is small when the angle of incidence is small. We can, therefore, obtain the first approximation to M and α from two observations of the difference of phase when the angle of incidence is not large. As the first approximation in such circumstances, we may neglect χ altogether, or give to it any small value that seems reasonable. Knowing M and α (approximately) we can then calculate $\theta' - \theta$, q/a, and γ' from the formulae above. Having obtained $\theta' - \theta$ by calculation and $\theta' - \theta - \chi$ from observation for any angle of incidence, we thus obtain χ for that incidence. Also we have

$$\tan \chi = q \sin \gamma / (1 + q \cos \gamma),$$

so that

$$1/a = (q/a) \sin (\gamma - \chi) \operatorname{cosec} \chi = (q/a) \sin (w + 2\alpha + \gamma' - \chi) \operatorname{cosec} \chi.$$

In this equation everything is known (approximately) except a and w, so that by using our knowledge of Δ at two angles of incidence, other than those employed for obtaining the approximate values of M and α, we obtain the unknown quantities a and w. Having thus obtained approximate values of the four optical constants, we can calculate χ in terms of them, and proceed if necessary to higher orders of approximation. If the experimental results give us the difference of phase for more than four angles of incidence, we may use the ordinary rules for finding the most probable values of the optical constants, and estimate the probable errors. The values of χ, for different angles of incidence, obtained in this way from Meslin's experiments on gold are given in the following table.

ϕ......	30°	40°	50°	60°	70°	74°	80°	85°
χ	0·009	0·017	0·027	0·041	0·055	0·061	0·051	0·027
Δ (theory)	0·056	0·103	0·176	0·270	0·423	0·5	0·663	0·815
Δ (exp.)	0·064	0·112	0·176	0·260	0·420	0·5	0·648	0·828
Difference	−0·008	−0·009	0	+0·010	+0·003	0	+0·015	−0·013

The table also compares the observed and calculated values for Δ, a comparison which is also brought out in Fig. 101, which represents the same results graphically.

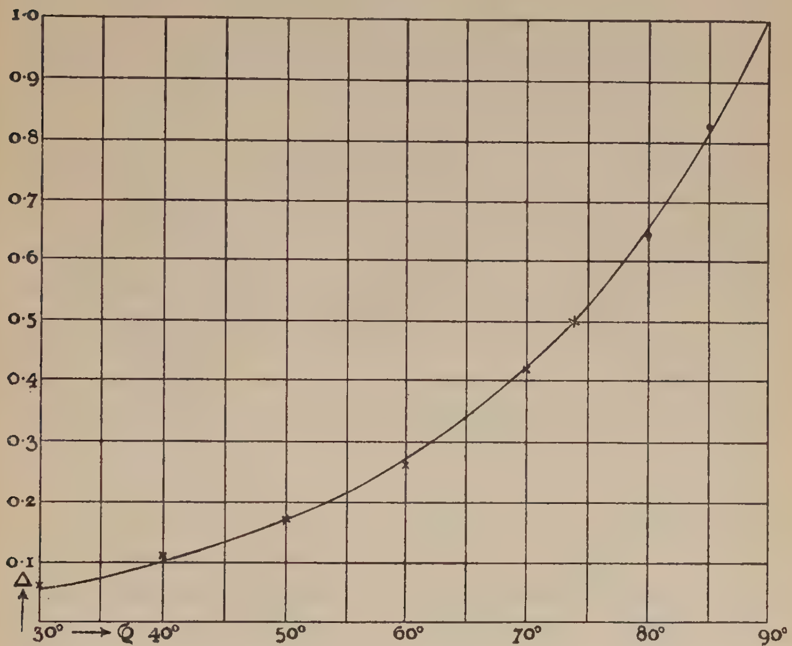

Fig. 101.

It has been supposed in all the formulae of this chapter that the medium into which reflection takes place is air. If this be not the case, the formulae must be somewhat modified. Thus, if the medium in contact with the metal be of refractive index μ_1, we have to replace M in the formulae by M/μ_1, keeping α as before. We have seen that for an abrupt transition the Principal Incidence is given by the formula $\sin^4\phi = M^2c^2\cos^2\phi$, which is approximately equivalent to $\sec\phi = M + M^{-1}(1-\frac{1}{2}\cos 2\alpha)$. If, however, there be a surface layer of transition we have

$$\sin^4\phi = M^2c^2\cos^2\phi + 2Mc\cos\phi\sin^2\phi\sin(\alpha+u)\tan\chi.$$

Since $\tan\chi$ is always small we may in the last term put

$$\sin^2\phi = Mc\cos\phi,$$

and we then get $\sin^4\phi = M^2c^2\cos^2\phi\,(1+2\kappa)$, where κ is a small quantity given by the formula $\kappa = \sin(\alpha+u)\tan\chi$. Thus the effect of the layer is to replace M^2 by $M^2(1+2\kappa)$ or M by $M(1+\kappa)$, while the effect of replacing air by a medium of refractive index μ_1 is to change M into M/μ_1. Hence, if there be a layer of transition between a medium μ_1 and a metal, the Principal Incidence will be determined by the equation

$$\sec\phi = M' + M'^{-1}(1 - \tfrac{1}{2}\cos 2\alpha), \text{ where } M' = M(1+\kappa)/\mu_1.$$

An increase in μ_1 will diminish M' and so diminish ϕ, the Principal Incidence. This agrees with Conroy's measurements of the Principal Incidences with gold and silver in contact with different media. Thus he found the Principal Incidences to be 71° 43′, 67° 39′, and 66° 36′ respectively for the reflection of yellow light from gold in air, in water, and in carbon bisulphide, and to be 74° 37′, 72° 15′, and 71° 39′ respectively for reflection from silver in air, in water, and in carbon tetrachloride. A change in μ_1 will also affect the Principal Azimuth (β). We have

$$\tan\beta = (1 + q\cos\gamma)\tan(\alpha+u)/2,$$

the angle u being given by the equation

$$\cot 2u = (M/\mu_1)^2 . \operatorname{cosec}^2\phi \operatorname{cosec} 2\alpha - \cot 2\alpha.$$

An increase in μ_1 will diminish $\cot 2u$ and therefore increase u, so that as a rule β will be slightly increased, although in some cases the increase of $\tan(\alpha+u)/2$ may be counterbalanced by the diminution of the factor $1 + q\cos\gamma$. This is also in accord with Conroy's measurements. He found the Principal Azimuths corresponding to the Principal Incidences given above to be 41° 4′, 41° 15′, and 41° 41′ for gold, and 43° 22′, 44° 9′, and 43° 40′ for silver.

CHAPTER X.

ON NEWTON'S RINGS FORMED BY METALLIC REFLECTION AND ON SOME OPTICAL PROPERTIES OF THIN METALLIC PLATES.

AFTER what has been done in the last chapter we are in a position to discuss some special problems presented by the propagation of light in metals. Of these one of the most interesting is that of the peculiarities of Newton's rings when formed by reflection from a metal. The main features of this phenomenon are thus described by Sir G. Stokes. "When Newton's rings are formed between a lens and a plate of metal, as the angle of incidence is increased the rings, which are at first dark-centred, disappear on passing the polarising angle of the glass, and then reappear white-centred, in which state they remain up to a grazing incidence, when they can no longer be followed. At a high incidence the first dark ring is much the most conspicuous of the series.

"To follow the rings beyond the limit of total internal reflection, we must employ a prism. When the rings formed between glass and glass are viewed in this way, we know that as the angle of incidence is increased the rings one by one open out, uniting with bands of the same respective orders which are seen beneath the limit of total internal reflection; the limit or boundary between total and partial reflection passes down beneath the point of contact, and the central spot is left isolated in a bright field.

"Now when the rings are formed between a prism with a slightly convex base and a plate of silver, and the angle of incidence is increased so as to pass the critical angle, if common light be used, in lieu of a simple spot we have a ring, which becomes more conspicuous at a certain angle of incidence

well beyond the critical angle, after which it rapidly contracts and passes into a spot.

"As thus viewed the ring is, however, somewhat confused. To study the phenomenon in its purity we must employ polarised light, or, what is more convenient, analyse the reflected light by means of a Nicol's prism. When viewed by light polarised in the plane of incidence, the rings show nothing remarkable. They are naturally weaker than with glass, as the interfering streams are so unequal in intensity. They are black-centred throughout, and, as with glass, they open out one after another on approaching the limit of total reflection and disappear, leaving the central spot isolated in the bright field beyond the limit. The spot appears to be notably smaller than with glass under like conditions.

"With light polarised perpendicularly to the plane of incidence, the rings pass from dark-centred to bright-centred on passing the polarising angle of the glass, and open out as they approach the limit of total reflection. The last dark ring to disappear is not, however, the first, but the second. The first, corresponding in order to the first bright ring within the polarising angle of the glass, remains isolated in the bright field, enclosing a relatively, though not absolutely bright spot. At the centre of the spot the glass and metal are in optical contact, and the reflection takes place accordingly, and is not total. The dark ring, too, is not absolutely black. As the angle of internal incidence increases by a few degrees, the dark ring undergoes a rapid and remarkable change. Its intensity increases till (in the case of silver) the ring becomes sensibly black, then it rapidly contracts, squeezing out, as it were, the bright central spot and forming itself a dark spot, larger than with glass, isolated in the bright field. When at its best it is distinctly seen to be fringed with colour, blue outside, red inside (especially the former). This rapid alteration taking place well beyond the critical angle is very remarkable. Clearly there is a rapid change in the reflective properties of the metal, which takes place, so to speak, in passing through a certain angle determined by a sine greater than unity.

"I have described the phenomenon with silver, which shows it best; but speculum-metal, gold, and copper show it very well, while with steel it is far less conspicuous."

In discussing the problems suggested by this description we shall employ the same method and notation as in Chapter V. It was there proved (p. 95) that the reflected beam is represented by $(r_1 + q^2 r_2)/(1 + q^2 r_1 r_2)$, where in general r_1, r_2, and q^2 are complex. Putting $r_1 = R_1 e^{i\theta_1}$, $r_2 = R_2 e^{i\theta_2}$, and $q^2 = Qe^{-i\psi}$, the intensity of the reflected light is

$$I = \frac{R_1^2 + Q^2 R_2^2 + 2QR_1 R_2 \cos(\theta_2 - \psi - \theta_1)}{1 + Q^2 R_1^2 R_2^2 + 2QR_1 R_2 \cos(\theta_2 - \psi + \theta_1)}.$$

It will be necessary to distinguish two cases—first, when the angle of incidence is less than the critical angle of the glass, and second when it is greater than that angle. In each case vibrations parallel and perpendicular to the plane of incidence must be dealt with separately.

If ϕ be the angle of incidence, ϕ' that of refraction, and μ_1 the refractive index of the glass, we have $\sin\phi' = \mu_1 \sin\phi$. The surface of separation being the plane $x = 0$, and the plane of xy being that of incidence, the components of the displacement are proportional to

$$e^{ip[t - (x\cos\phi' + y\sin\phi')/c]}.$$

Putting $x = c_1$, the thickness of the film of air, we see that

$$q = e^{-ipc_1\cos\phi'/c} = e^{-i2\pi c_1/\lambda \,.\, \cos\phi'},$$

where λ is the wave-length in air. Below the critical angle ϕ' is real, so that we have $Q = 1$, and $\psi = 2\pi c_1/\lambda \,.\, \cos\phi'$. Hence

$$I = \frac{a + b\cos(\theta_2 - \psi - \theta_1)}{a_1 + b\cos(\theta_2 - \psi + \theta_1)},$$

where $a = R_1^2 + R_2^2$, $b = 2R_1 R_2$, and $a_1 = 1 + R_1^2 R_2^2$. For a given angle of incidence R_1, R_2, θ_1 and θ_2 are constants, and the only variable in the expression for I is ψ, which depends on the thickness of the film of air between the glass and the metal. Differentiating I with respect to ψ, we find that it is stationary when

$$(a - a_1)\cos\theta_1 \sin(\theta_2 - \psi) + (a + a_1)\sin\theta_1 \cos(\theta_2 - \psi) \\ + b\sin 2\theta_1 = 0.$$

Putting
$$\frac{a_1 + a}{a_1 - a}\tan\theta_1 = \tan\beta,$$

and
$$\frac{b\sin 2\theta_1}{\sqrt{(a_1 - a)^2 + 4aa_1\sin^2\theta_1}} = \sin\gamma,$$

we get $\sin(\psi+\beta-\theta_2)=\sin\gamma$, so that $\psi+\beta-\theta_2=n\pi+(-1)^n.\gamma$, and $2\pi c_1/\lambda=\sec\phi'[n\pi+(-1)^n\gamma+\theta_2-\beta]$. By giving different integral values to n we see that there will be rings of maximum and minimum intensity.

Formulae suitable for the calculation of all the quantities R_1, R_2, θ_1, and θ_2 have already been obtained. It has been seen that they all depend to some extent on the character of the layer of transition from glass to air and from air to metal. The influence of the layer on the phenomena now under discussion is, however, very slight, and we shall obtain an approximation sufficiently accurate for present purposes by regarding the transition as abrupt. In that case r_1 and r_2 are given by Fresnel's formulae, and the modifications thereof obtained by making the refractive index complex for reflection from metal.

For light polarised at right angles to the plane of incidence we have

$$r_1=-\tan(\phi-\phi')/\tan(\phi+\phi')=\tan(\phi'-\phi)/\tan(\phi'+\phi).$$

When ϕ lies between zero and the polarising angle $\tan^{-1}1/\mu_1$, this gives $R_1=\tan(\phi'-\phi)/\tan(\phi'+\phi)$ and $\theta_1=0$, while when ϕ is greater than the polarising angle it gives

$$R_1=-\tan(\phi'-\phi)/\tan(\phi'+\phi),\quad\text{and}\quad\theta_1=\pi.$$

In both cases $\beta=\gamma=0$, and I is stationary when $\psi-\theta_2=n\pi$. The maximum value of I is $(a+b)/(a_1+b)=(R_1+R_2)^2/(1+R_1R_2)^2$, and the minimum is $(a-b)/(a_1-b)=(R_1-R_2)^2/(1-R_1R_2)^2$. When ϕ is less than the polarising angle I is a maximum when n is even and a minimum when n is odd; but this state of affairs is reversed when ϕ is greater than the polarising angle. At the polarising angle $b=0$, so that I is constant and there are no rings.

For the determination of the radii of the rings and their intensity nothing remains but the calculation of the constants R_1, R_2, and θ_2. For this purpose we shall take $\mu_1=1{\cdot}596$, which corresponds to a polarising angle of 32° 1′, and a critical angle of 38° 47′. For the metals we shall take Drude's estimates of μ_0 and a for sodium light reflected from silver and steel respectively, viz.: $\mu_0=0{\cdot}18$ and $a=3{\cdot}67$ in the case of silver, and $\mu_0=2{\cdot}41$ and $a=3{\cdot}40$ in the case of steel. The values of R_2 and θ_2 are obtained from the formulae of p. 242.

With the aid of these numbers there is no difficulty in calculating the intensity I at any point. For our purposes, however,

ϕ...	0°	10°	20°	30°	35°	38°	38° 15′	38° 30′	38° 46′
R_1	·230	·220	·175	·052	·121	·454	·526	·625	·881
R_2 silver	·976	·975	·971	·962	·952	·953	·957	·965	·989
R_2 steel	·764	·759	·709	·655	·566	·517	·545	·608	·859
θ_2 silver	329° 35′	328° 18′	323° 39′	310° 24′	290° 10′	249° 0′	236° 52′	223° 38′	192° 26′
θ_2 steel	337° 26′	336° 41′	333° 5′	322° 26′	303° 18′	251° 20′	239° 17′	223° 46′	191° 42′

it will be sufficient to obtain the maximum and minimum values, and the intensity at the centre. The maxima and minima for silver and steel are given in the following table.

ϕ......	0°	10°	20°	30°	35°	38°	38° 15′	38° 30′	38° 46′
I (max.) silver	·969	·967	·958	·932	·926	·962	·974	·982	·998
I (min.) silver	·924	·923	·918	·917	·882	·773	·753	·732	·704
I (max.) steel	·715	·703	·618	·468	·413	·619	·692	·798	·981
I (min.) steel	·420	·419	·372	·390	·228	·007	·001	·001	·008

The value of the intensity at the centre cannot be derived from these results, for at that point there is no air space between the glass and metal, so that we have direct reflection from the metal. The intensity in this case is most simply obtained from the formulae of p. 242, by replacing μ_0 and a by μ_0/μ_1 and a/μ_1 respectively. In this way we get the following values of the intensity for different values of ϕ ranging from zero to 90°.

ϕ...	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
I (silver)	·931	·930	·929	·924	·918	·910	·905	·896	·925	1
I (steel)	·442	·438	·425	·406	·362	·316	·266	·215	·346	1

The size of the rings is easily obtained as on p. 104. We have seen that at any point on a ring of maximum or minimum intensity $\psi - \theta_2 = n\pi$. Also we have $\psi = 2\pi c_1/\lambda \cos\phi' = \pi\kappa_1 r^2/\lambda \, . \cos\phi'$, where λ is the wave length in the glass, not in the air. Hence $\rho = r\sqrt{\kappa_1/\lambda} = \sqrt{(n+\theta_2/\pi)\sec\phi'}$. By putting $n = -1, 0, 1, 2, 3$, etc., we get the radii of the successive rings. For the rings of high order the radii are very nearly proportional to the square roots of the natural numbers. The following tables give the values of ρ_1, ρ_2, ρ_3 and ρ_4 corresponding to the first four rings, for different values of ϕ ranging from zero to the critical angle.

ϕ	Silver				Steel			
	ρ_1	ρ_2	ρ_3	ρ_4	ρ_1	ρ_2	ρ_3	ρ_4
0°	0·912	1·35	1·68	1·96	0·935	1·37	1·70	1·97
10°	0·925	1·38	1·71	2·00	0·952	1·40	1·73	2·01
20°	0·976	1·47	1·83	2·13	1·01	1·53	1·88	2·17
30°	1·10	1·69	2·13	2·48	1·15	1·72	2·15	2·51
35°	1·23	2·00	2·55	3·00	1·31	2·05	2·58	3·03
38°	1·41	2·72	3·58	4·26	1·43	2·73	3·59	4·27
38° 15′	1·44	2·93	3·89	4·66	1·44	2·95	3·90	4·67
38° 30′	1·463	3·31	4·45	5·35	1·45	3·31	4·45	5·35
38° 46′	1·496	5·89	8·19	9·98	1·45	5·88	8·19	9·97
38° 47′	1·5	∞	∞	∞	1·45	∞	∞	∞

From these results it appears that the rings open out as the angle of incidence increases. The expansion of all the rings except the first is very rapid when nearing the critical angle, and the radius of each increases to infinity when the critical angle is reached. With the first ring, however, there is very little change as the critical angle is approached, and the limiting value of $r\sqrt{\kappa_1/\lambda}$ is 1·5 for silver and 1·45 for steel. Our formulae cannot, in strictness, be applied when ϕ is exactly equal to the critical angle,

for then $|q^2 r_2 r_3| = 1$, and the fundamental series on p. 95 loses its convergence. We may, however, approach the critical angle as near as we wish. The angle θ_2 is obtained from the formula of p. 242, viz.: $\tan\theta_2 = -2Mc\cos\phi'\sin(\alpha - u)/(M^2\cos^2\phi' - c^2)$. When ϕ' is nearly 90°, $\tan\theta_2$ is very small, and we have

$$\theta_2 - \pi = \tan(\theta_2 - \pi), \text{ approximately,}$$
$$= \tan\theta_2 = -2Mc\cos\phi'\sin(\alpha - u)/(M^2\cos^2\phi' - c^2).$$

Hence $(\theta_2 - \pi)\sec\phi' = 2Mc\sin(\alpha - u)/c^2$ in the limit. For the two metals, silver and steel, M^2 is 13·5 and 17·37 respectively, so that the error in neglecting M^{-4} and higher powers of M^{-2} is very small. To this order of approximation we find from the formulae of p. 242 that, when $\phi' = 90°$, we have

$$2Mc\sin(\alpha - u)/c^2 = 2a(1 - 1/2M^2).$$

Hence $\rho_1 = \sqrt{\sec\phi'(\theta_2/\pi - 1)} = \sqrt{(1 - 1/2M^2)\,2a/\pi}$ which gives the limiting values recorded above for silver and steel.

From the formula $r\sqrt{\kappa_1/\lambda} = \sqrt{(n + \theta_2/\pi)\sec\phi'}$, we see that r is a function of λ so that, unless homogeneous light be employed, the rings will be coloured. ϕ' and consequently θ_2 are functions of λ, but the change in r due to this fact is usually small, so that approximately r varies as the square root of the wave length. The following table gives the values of ρ corresponding to the

		$\phi = 0°$		$\phi = 20°$		$\phi = 38°$	
		Silver	Steel	Silver	Steel	Silver	Steel
First bright ring	ρ_h	0·741	0·760	0·793	0·821	1·15	1·16
	ρ_d	0·912	0·935	0·976	1·01	1·41	1·43
	ρ_a	1·04	1·06	1·11	1·15	1·60	1·62
First dark ring	ρ_h	1·10	1·11	1·19	1·24	2·21	2·22
	ρ_d	1·35	1·37	1·47	1·53	2·72	2·73
	ρ_a	1·53	1·56	1·67	1·74	3·08	3·10
Second bright ring	ρ_h	1·37	1·38	1·49	1·53	2·91	2·92
	ρ_d	1·68	1·70	1·83	1·88	3·58	3·59
	ρ_a	1·91	1·93	2·06	2·13	4·06	4·08
Second dark ring	ρ_h	1·59	1·60	1·73	1·76	3·46	3·47
	ρ_d	1·96	1·97	2·13	2·17	4·26	4·27
	ρ_a	2·22	2·24	2·42	2·46	4·84	4·85

lines A, D, and H in the spectrum, calculated on the hypothesis that ρ is proportional to the square root of λ, an hypothesis that will serve for illustrative purposes.

The general effect of this dependence of the radius on the wave length is that each bright ring changes in colour from violet on the inside to red on the outside, whereas in the dark rings the order of colours is reversed. In the case of silver, however, the difference between the maximum and the minimum intensities is nowhere great, so that the rings are not sufficiently conspicuous to show the colours well. Another effect, well illustrated by the tables above, is that, for all the rings except the first, the red of one ring overlaps the violet of the next. This diminishes the sharpness of the rings, and makes the first appear more conspicuous than the others.

Most of these matters are illustrated graphically in the accompanying figures, which embody the results obtained in Chapter V.,

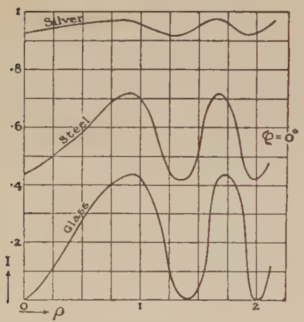

Fig. 102.

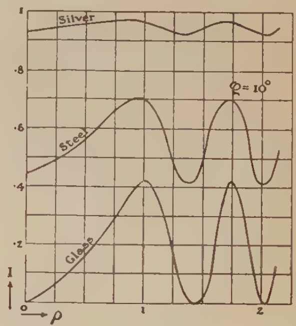

Fig. 103.

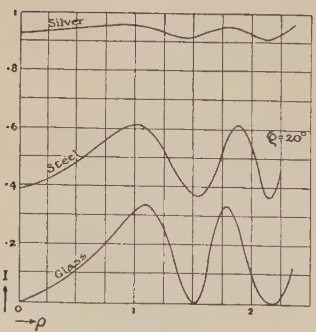

Fig. 104.

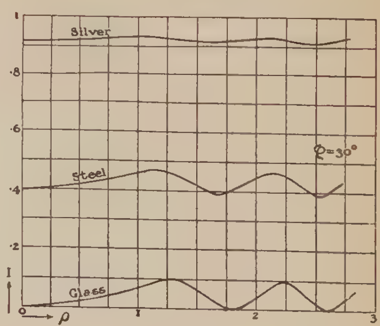

Fig. 105.

when the metal is replaced by glass of the same refractive index as the first piece.

Figs. 102, 103, 104 and 105 represent the intensities as far as the second dark ring for various angles of incidence below the polarising angle, viz. $\phi = 0°$, $10°$, $20°$, and $30°$ respectively. It is evident from the figures that the rings are all dark-centred, that they gradually expand as ϕ increases, and that they are much less distinct with silver than with steel or glass. The rings are most distinct with glass, but the difference between that and steel is not very marked. Fig. 105 shows that as the polarising angle is approached the rings rapidly disappear. After passing that angle the rings pass from dark-centred to light-centred in the case of the metals; but with glass they remain dark-centred throughout. This is clearly brought out in Figs. 106, 107, 108, and 109 which follow.

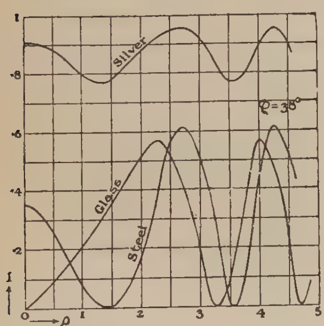

Fig. 106.

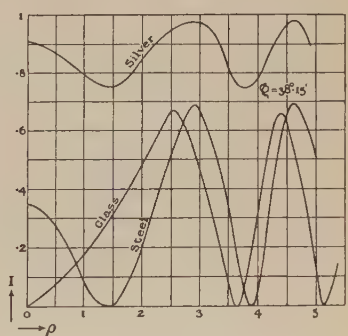

Fig. 107.

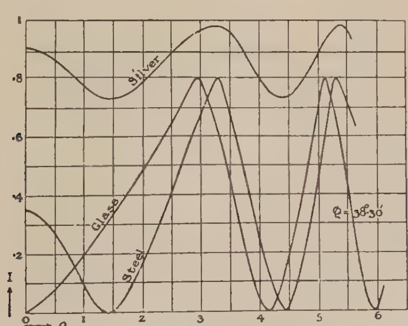

Fig. 108.

The rings are still very much more distinct with glass and steel than with silver. There is a close resemblance between the

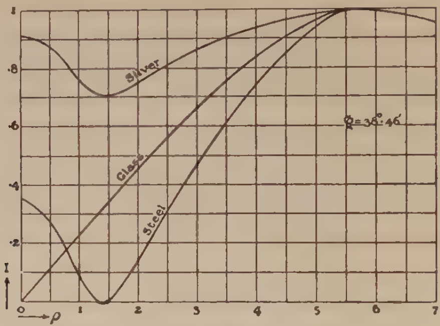

Fig. 109.

rings with glass and with steel, the main difference being that the rings with steel are bright-centred, but the brightness at the centre is not very appreciable compared with that at the rings. The rings with steel are slightly more distinct than those with glass and slightly larger. After passing the polarising angle the rings have all expanded so much that it is convenient to use a smaller scale to represent the radius, and in the figures 106—109, the scale of ρ is half that in the earlier figures 102—105.

The colour effects are indicated in Figs. 110 and 111, the first of which deals with the case where $\phi = 0$, and the second with that where $\phi = 38°$, i.e. beyond the polarising angle. The dotted curve represents the intensity for red light, and the continuous curve that for violet.

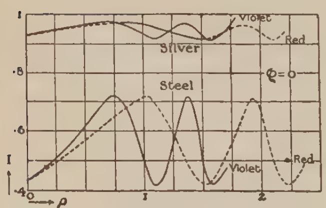

Fig. 110.

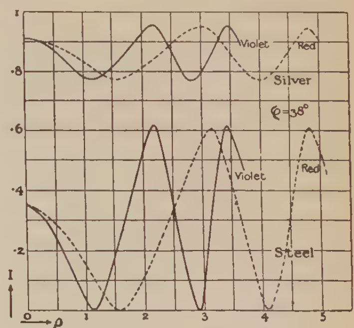

Fig. 111.

The discussion of the phenomena when the incident light is polarised parallel to the plane of incidence may be conducted on exactly similar lines. In this case we have

$$r_1 = \sin(\phi' - \phi)/\sin(\phi' + \phi) = R_1 \text{ and } \theta_1 = 0.$$

The maxima and minima are given by the same formulae as before (p. 266), and occur as before when $\psi - \theta_2 = n\pi$. The rings are dark-centred throughout instead of passing from dark-centred to bright-centred as with light polarised at right angles to the plane of incidence. The values of R_1, R_2, and θ_2 are as follows:

ϕ......	0°	10°	20°	30°	35°	38°	38° 46′
R_1	·230	·243	·283	·393	·530	·743	·952
R_2 (silver)	·976	·976	·980	·985	·992	·996	·999
R_2 (steel)	·764	·773	·799	·852	·911	·952	·999
θ_2 (silver)	329° 35′	330° 48′	334° 36′	341° 49′	349° 52′	354° 25′	360°
θ_2 (steel)	337° 26′	338° 21′	341° 12′	346° 23′	353° 0′	355° 48′	360°

From these results we derive the following values for the intensity at the bright and dark rings.

ϕ......	0°	10°	20°	30°	35°	38°	38° 46′
I (max.) silver	·969	·971	·976	·987	·996	·999	1
I (min.) silver	·924	·924	·929	·935	·949	·959	1
I (max.) steel	·715	·731	·779	·869	·945	·987	1
I (min.) steel	·420	·426	·445	·476	·542	·551	1

The value of the intensity at the centre, obtained as on p. 267, is as follows:

ϕ...	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
I (silver)	·931	·932	·936	·941	·949	·957	·967	·978	·989	1
I (steel)	·443	·449	·467	·484	·541	·600	·674	·765	·873	1

For the radii of the rings we have the following values:

ϕ	Silver				Steel			
	ρ_1	ρ_2	ρ_3	ρ_4	ρ_1	ρ_2	ρ_3	ρ_4
0°	0·912	1·35	1·68	1·96	0·935	1·37	1·70	1·97
10°	0·934	1·39	1·72	2·00	0·957	1·40	1·73	2·01
20°	1·01	1·49	1 85	2·15	1·03	1·50	1·86	2·16
30°	1·22	1·99	2·19	2·54	1·24	1·79	2·20	2·55
35°	1·53	2·20	2·71	3·13	1·55	2·21	2·71	3·14
38°	2·29	3·26	3·91	4·63	2·30	3·27	4·01	4·63
38° 47′	∞	∞	∞	∞	∞	∞	∞	∞

The graphical representation of these results is undertaken in the next three figures. It appears from these that the rings with silver are very indistinct. With steel, on the other hand, the rings begin, when ϕ is small, by being almost as marked as with glass. They remain fairly distinct as ϕ increases, but the rings with glass gain in intensity much more conspicuously than those with steel. In all cases the rings are dark-centred; but, as will be seen from Fig. 114, as the critical angle is approached the difference between the intensity at the centre and at the first maximum is very slight in the case of the metals, so that there appears to be a bright spot at the centre.

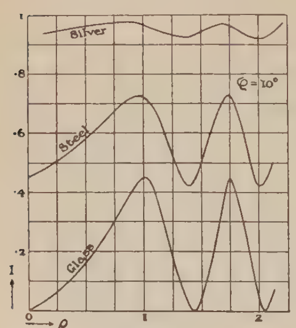

Fig. 112.

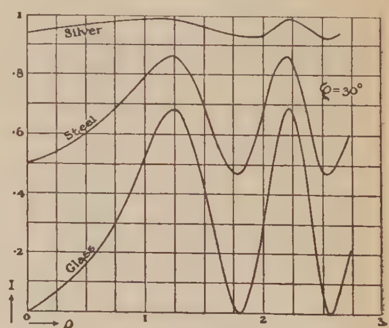

Fig. 113.

When a glass prism is employed and the incidence is beyond the critical angle, we must make use of the formulae obtained in

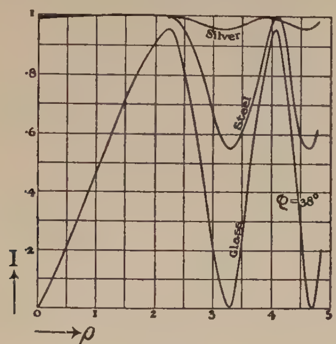

Fig. 114.

Chapter III. for the change of amplitude and phase in the case of total reflection. When the incident light is polarised at right angles to the plane of incidence we have seen (p. 56) that $r_1 = e^{i(2\alpha_1 - \pi)}$, where

$$\tan \alpha_1 = \mu_1^2 \sec \phi \sqrt{\sin^2 \phi - 1/\mu_1^2}.$$

Hence $R_1 = 1$, and $\theta_1 = 2\alpha_1 - \pi$. To find the values of R_2 and θ_2 we must consider the reflection at the metal of the "surface waves" due to total reflection. In the air we have

$$\zeta = e^{ip[t-(nx+y \sin \phi)\mu_1/c]} - r_2 e^{ip[t+(nx-y \sin \phi)\mu_1/c]},$$

where
$$n = -i\sqrt{\sin^2 \phi - 1/\mu_1^2},$$

the first term corresponding to the incident and the second to the reflected wave. In the metal we have similarly

$$\zeta = s e^{ip[t-(mx+y \sin \phi)\mu_1/c]}.$$

In order to satisfy the dynamical equations in the metal we must have

$$(m^2 + \sin^2 \phi)\mu_1^2 = \mu^2 = (\mu_0 - ia)^2,$$

so that m is always complex, of the form $m_1 - in_1 = Ne^{-i\gamma}$. The boundary conditions require the continuity of ζ and of g/μ^2, i.e. of ζ and of $\dfrac{1}{\mu^2}\dfrac{\partial \zeta}{\partial x}$. Hence we must have $1 - r_2 = s$, and

$$n(1 + r_2) = ms/\mu^2,$$

and these give

$$r_2 = \frac{m - n\mu^2}{m + n\mu^2} = \frac{m_1 + p_1 M^2 \sin 2\alpha - i(n_1 + p_1 M^2 \cos 2\alpha)}{m_1 - p_1 M^2 \sin 2\alpha - i(n_1 - p_1 M^2 \cos 2\alpha)},$$

where $$\mu_0 - i a = M e^{-i\alpha},$$

and $$p_1 = \sqrt{\sin^2 \phi - 1/\mu_1^2}.$$

Thus R_2 and θ_2 are determined from the equations

$$R_2^2 = (1 + x)/(1 - x),$$

where

$$x = \frac{2p_1 M^2 (m_1 \sin 2\alpha - n_1 \cos 2\alpha)}{m_1^2 + n_1^2 + p_1^2 M^4} = \frac{2p_1 M^2 N \sin(2\alpha - \gamma)}{N^2 + p_1^2 M^4},$$

and

$$\tan \theta_2 = \frac{2p_1 M^2 (m_1 \cos 2\alpha + n_1 \sin 2\alpha)}{m_1^2 + n_1^2 - p_1^2 M^4} = \frac{2p_1 M^2 N \cos(2\alpha - \gamma)}{N^2 - p_1^2 M^4}.$$

We thus derive the following table:

ϕ......	38° 47′	40°	45°	50°	60°	70°	80°
$\theta_1 + 180°$	0	51° 16′	99° 32′	120° 24′	143° 40′	158° 8′	169° 46′
R_2 (silver)	1	2·04	7·37	35·4	6·78	4·48	4·09
R_2 (steel)	1	1·40	1·93	2·11	2·14	2·08	2·04
θ_2 (silver)	180°	177° 41′	168° 56′	75° 11′	10° 29′	6° 50′	6° 14′
θ_2 (steel)	180°	156° 50′	124° 54′	106° 7′	84° 31′	73° 46′	68° 38′

We have seen (p. 265) that the intensity is given by the formula

$$I = \frac{R_1^2 + Q^2 R_2^2 + 2QR_1R_2 \cos(\theta_2 - \psi - \theta_1)}{1 + Q^2 R_1^2 R_2^2 + 2QR_1R_2 \cos(\theta_2 - \psi + \theta_1)}.$$

In the present case we have $R_1 = 1$, $\psi = 0$ and

$$Q = e^{-4\pi p_1 c_1/\lambda} = e^{-2\pi\kappa_1 p_1 r^2/\lambda},$$

where λ is the wave length in the glass, and r is the distance from the centre. We thus have

$$I = \frac{1 + Q^2 R_2^2 + 2QR_2 \cos(\theta_2 - \theta_1)}{1 + Q^2 R_2^2 + 2QR_2 \cos(\theta_2 + \theta_1)}.$$

The numerator is the square of the resultant of two vectors

1 and QR_2 inclined at an angle $\theta_2 - \theta_1$. This resultant will be least when $\theta_2 - \theta_1 = 180°$, and its least value will be zero if $1 = QR_2$. Hence, if we are seeking for places of absolute blackness we must take $\theta_2 - \theta_1 = 180°$ and $1 = QR_2$. The former equation, if it can be satisfied at all, will confine us to a particular angle of incidence ϕ; the latter will tie us to a ring of definite radius. It may be, however, that these conditions cannot both be satisfied, but there may still be, at different angles of incidence, a *dark* ring where the intensity is a minimum. For a given angle of incidence R_2, θ_2 and θ_1 are fixed, so that to find the position of the dark ring we have to differentiate I with respect to Q. In this way we learn that I is stationary when $Q = 1/R_2$. The corresponding value of I is a *minimum* if $\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)$ be negative, which is always the case since I cannot be greater than unity. The minimum value of I is $\dfrac{1 + \cos(\theta_1 - \theta_2)}{1 + \cos(\theta_1 + \theta_2)}$. The black ring occurs when $\theta_2 = \theta_1 + 180°$, and its position is determined by drawing graphs of the functions θ_2 and $\theta_1 + 180°$, and finding where they intersect. This is done in Fig. 115, from which it appears that

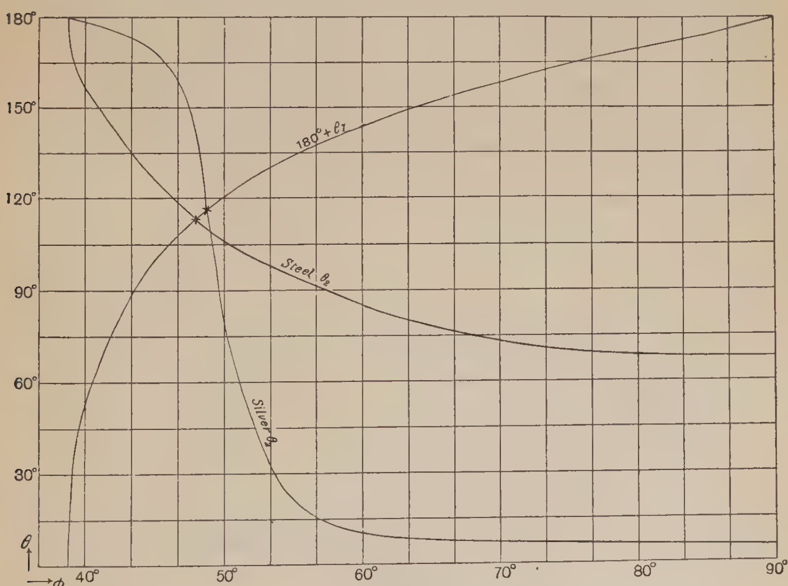

Fig. 115.

with silver the black ring corresponds to an incidence of $\phi = 48° 40'$, and with steel to $\phi = 47° 55'$.

The figure also shows that when the incidence is near that giving an absolutely black ring, the angle $\theta_1 - \theta_2$ varies very much more rapidly with silver than with steel, so that the change of intensity at the dark ring will be much more marked with the first metal than with the second. The distinctness of a ring depends not only on the intensity of the light there, but also on the difference of intensities at the ring and near its centre. Very near the centre we have $Q = 1$, so that the intensity is

$$I = \frac{1 + R_2^2 + 2R_2 \cos(\theta_1 - \theta_2)}{1 + R_2^2 + 2R_2 \cos(\theta_1 + \theta_2)}.$$

The following table gives the minimum value of I, i.e. its value at the dark ring, as well as that near the centre, for different incidences.

ϕ......	38° 47′	40°	45°	50°	60°	70°	80°
I (min.) silver	1	·962	·631	·138	·880	·978	·980
I (min.) steel	1	·674	·056	·018	·292	·558	·783
I (near centre) silver	1	·966	·900	·908	·946	·987	·988
I (near centre) steel	1	·689	·164	·161	·402	·623	·813

The radius of the dark ring is obtained from the formula

$$R_2 = 1/Q = 1/q^2 = e^{2\pi p_1 \kappa_1 r^2/\lambda},$$

which gives

$$\rho = r\sqrt{\kappa_1/\lambda} = \sqrt{\log_e R_2^2/4\pi}\,\sqrt{p_1}.$$

This gives the following values of ρ for different incidences with silver and steel.

ϕ......	40°	45°	50°	60°	70°	80°
ρ (silver)	·546	·745	·925	·628	·534	·506
ρ (steel)	·380	·427	·423	·396	·374	·360

Fig. 116 embodies these results, and shows in a graphic way how the ring expands and contracts as the angle of incidence increases.

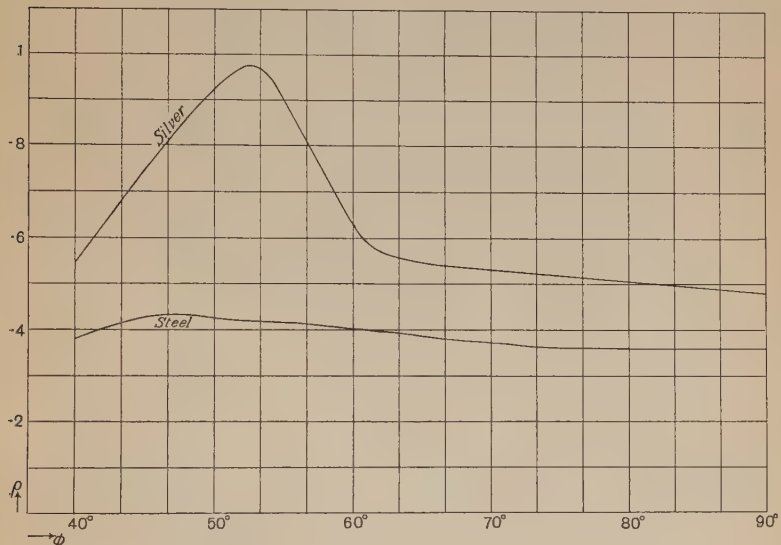

Fig. 116.

It will be observed that with silver the ring contracts very rapidly, shortly after the incidence corresponding to the black ring has been reached, and that the contraction is not nearly so marked in the case of steel. Another difference between the behaviour of the two metals is that the difference between the intensity at the ring and that near the centre is much greater for silver than for steel. This adds greatly to the distinctness of the ring, which with steel is, as a rule, scarcely distinguishable from the dark patch that it encloses. Some of the results in the above tables are illustrated in Figs. 117, 118, and 119, in which the ordinates represent the blackness $B=1-I$ for different angles of incidence. When the metal is replaced by glass the values of B are given on p. 110, and these are used in the figures in order to represent to the eye the differences in the phenomena presented by glass, steel and silver.

Fig. 117 represents the state of affairs when $\phi = 45°$. The ring with silver is quite conspicuous, but the blackness is much

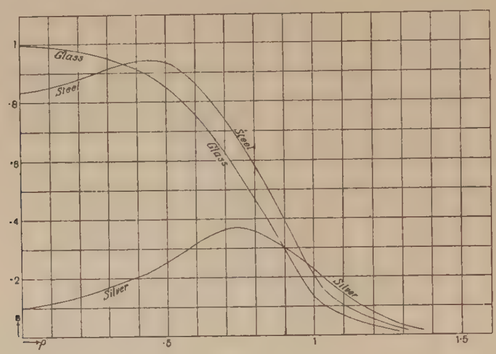

Fig. 117.

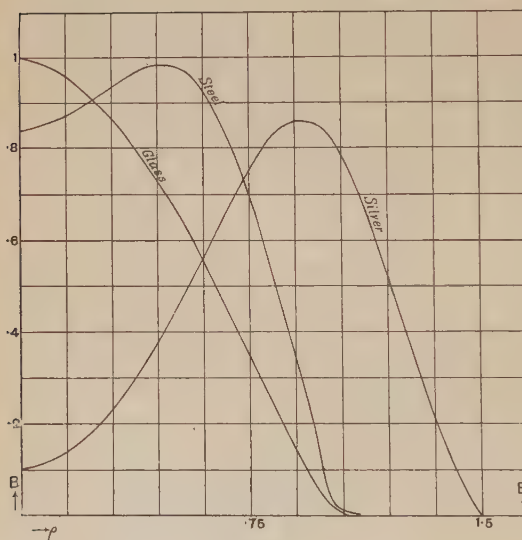

Fig. 118.

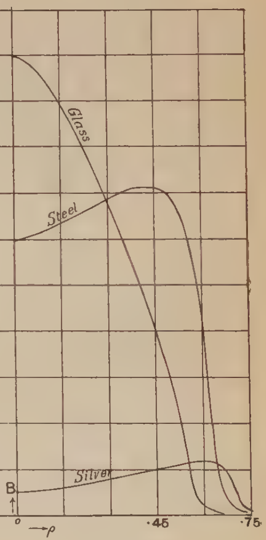

Fig. 119.

less than with steel. The centre of the ring with steel is, however, almost as dark as the ring, so that the general effect is rather that of a dark central patch very like that with glass. In the next figure the angle of incidence has increased by five degrees. The ring with silver is now much more pronounced,

being larger and more intense; but there is no very marked change with the ring in the case of steel. Fig. 119 represents the change that has taken place when $\phi = 60°$. The ring with silver has contracted so much as to squeeze out its bright centre almost completely, and both rings now present the appearance of a dark spot at the centre, larger than with glass, but less intense.

As the radii of the rings depend on λ and on the optical constants, they will be different for different colours. The general character of the chromatic effects is indicated in Fig. 120, where the intensities are represented for different colours.

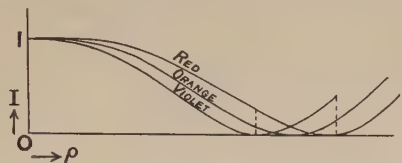

Fig. 120.

It appears from the figure that the black ring is coloured red on the inside and violet on the outside. The intensity of the violet on the outside is greater than that of the red on the inside, so that the outside colour is more marked, as was pointed out by Stokes in the description with which this chapter opened.

Turning to the case where the incident light is polarised parallel to the plane of incidence, we make use of the results obtained on p. 57. We have $r_1 = e^{2i\alpha'}$, where

$$\tan \alpha' = p_1 \sec \phi = \tan \alpha_1 / \mu_1^2,$$

so that $R_1 = 1$ and $\theta_1 = 2\alpha'$. To find the values of R_2 and θ_2 we have

$$\zeta' = e^{ip\,[t-(nx+y\sin\phi)\,\mu_1/c]} + r_2 e^{ip\,[t+(nx-y\sin\phi)\,\mu_1/c]}$$

in the air, and

$$\zeta' = s e^{ip\,[t-(mx+y\sin\phi)\,\mu_1/c]}$$

in the metal, where n and m have the same values as before. The boundary conditions give $1 + r_2 = s$, and $n(1 - r_2) = ms$, so that $r_2 = (n - m)/(n + m)$. Thus R_2 and θ_2 are determined from the equations $R_2^2 = (1 - x)/(1 + x)$, where

$$x = \frac{2p_1 N \sin\gamma}{N^2 + p_1^2}, \text{ and } \tan\theta_2 = \frac{2p_1 N \cos\gamma}{N^2 - p_1^2}.$$

The values derived from these formulae are as follows:

ϕ	38° 47′	40°	45°	50°	60°	70°	80°
θ_1	0	21° 20′	49° 46′	68° 54′	100° 12′	127° 36′	154° 18′
R_2 (silver)	1	·951	·895	·860	·820	·796	·786
R_2 (steel)	1	·972	·938	·917	·891	·874	·865
θ_2 (silver)	180°	180° 8′	182° 55′	183° 49′	184°	185° 31′	185° 49′
θ_2 (steel)	180°	180° 39′	181° 27′	181° 56′	182° 33′	182° 55′	183° 8′

We have seen that the intensity I is stationary when $Q = 1/R_2$. In the present case, since $\sin \gamma$ is positive, x cannot be negative, so that R_2 cannot be greater than unity. Hence as θ cannot be greater than unity, I is nowhere stationary. So that there can be no rings. The intensity is least near the centre, where its value is given by putting $Q = 1$, so that

$$I = \frac{1 + R_2^2 + 2R_2 \cos(\theta_1 - \theta_2)}{1 + R_2^2 + 2R_2 \cos(\theta_1 + \theta_2)}.$$

The intensity increases rapidly with the distance from the centre, and the appearance presented is very similar to that when the metal is replaced by glass, except that there is not absolute blackness at the centre. The following table compares the intensities at different distances from the centre in the case of glass, silver and steel respectively.

ρ		0	0·5	1	2
$\phi = 40°$	Q	1·000	0·597	0·127	0·0003
	I (glass)	0·000	0·339	0·919	0·9997
	I (silver)	0·978	0·991	0·998	0·9997
	I (steel)	0·886	0·953	0·997	0·9997
$\phi = 60°$	Q	1·000	0·391	0·023	3×10^{-7}
	I (glass)	0·000	0·197	0·914	1·000
	I (silver)	0·891	0·930	0·995	1·000
	I (steel)	0·929	0·951	0·998	1·000

These results are represented in Fig. 121, where the ordinates correspond to the blackness $1-I$. It will be seen that there is little difference between the phenomena as presented by the two

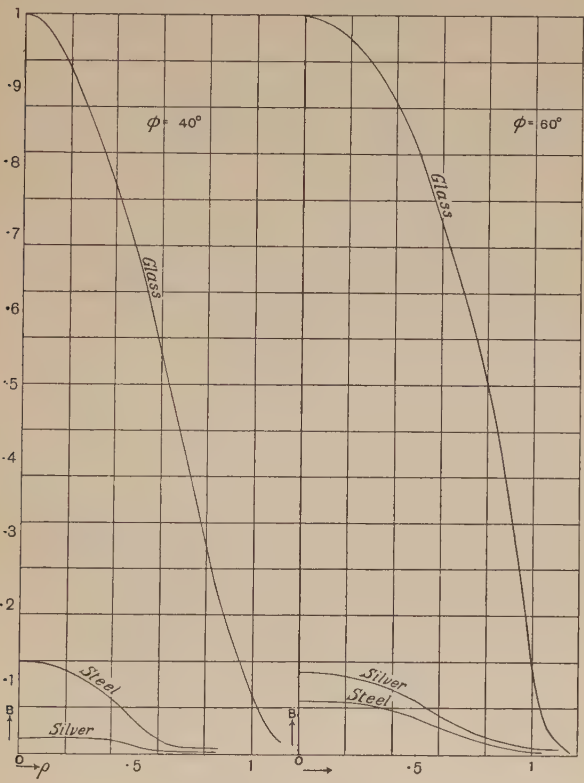

Fig. 121.

metals silver and steel, and that with each of these the intensity of the black spot is much less marked than in the case of glass.

It thus appears that all the peculiarities of Newton's rings formed by metallic reflection may be explained by means of the principles laid down in the last chapter. As another example of the use of these principles we shall discuss some of the optical properties of thin metallic plates. Adopting the same notation as before, the reflected and transmitted beams are represented by

$r(1-q^2)/(1-q^2r^2)$ and $q(1-r^2)/(1-q^2r^2)$ respectively. The wave first refracted into the metal is $se^{ip[t-(bx+y\sin\phi)/c]}$, where

$$b^2+\sin^2\phi=\mu^2.$$

Hence, if c_1 be the thickness of the plate, we have

$$q=e^{-ipbc_1/c}=e^{-i2\pi bc_1/\lambda},$$

where λ is the wave length of the incident light. Putting

$$\mu=\mu_0-ia=Me^{-ia},$$

we have $b=me^{-i\chi}$ where

$$m^4=M^4+\sin^4\phi-2M^2\cos 2\alpha\sin^2\phi,$$

and $$\tan 2\chi=M^2\sin 2\alpha/(M^2\cos 2\alpha-\sin^2\phi).$$

We thus get $$q^2=Qe^{-i\psi},$$

where $$Q=e^{-4\pi c_1/\lambda\,.\,m\sin\chi},$$

and $$\psi=4\pi c_1/\lambda\,.\,m\cos\chi.$$

These relations suffice to determine Q in terms of the optical constants of the metal and the thickness of the plate. The quantity r is also a complex, given by the formulae on p. 241.

If the incident wave be of unit amplitude, then the reflected beam is represented by $R_1e^{i\theta_1}$, where R_1 is the amplitude and θ_1 the change of phase produced by reflection. We have

$$R_1e^{i\theta_1}=\frac{r(1-q^2)}{1-q^2r_2}=\frac{Re^{i\theta}(1-Qe^{-i\psi})}{1-QR^2e^{i(2\theta-\psi)}}.$$

We thus get

$$R_1^2=R^2\,.\,\frac{1+Q^2-2Q\cos\psi}{1+Q^2R^4-2QR^2\cos(\psi+\gamma)},$$

and $\theta_1=\theta+\Phi'-\Phi$, where $\theta=\pi-\gamma/2$,

$$\tan\Phi=\frac{QR^2\sin(\psi+\gamma)}{1-QR^2\cos(\psi+\gamma)},$$

and Φ' is obtained from Φ by putting $R=1$ and $\theta=0$. As the thickness of the plate increases, Q approaches the limit zero, so that R_1 and θ_1 approach asymptotically to the values R and θ, as is to be expected. The values of R_1^2 and θ_1 are given below for silver in the case of direct incidence, the optical constants

employed being $a = 3{\cdot}67$ and $\mu_0 = 0{\cdot}18$, and θ_1 being expressed as a fraction of the half wave length.

c_1/λ	Indef. small	0·001	0·005	0·01	0·02	0·05	0·1	0·2	∞
R_1^2	0	·002	·061	·163	·430	·826	·938	·952	·952
θ_1	·529	·542	·591	·641	·717	·806	·829	·831	·831

These results are represented graphically in the curves marked 1 in Figs. 122 and 123.

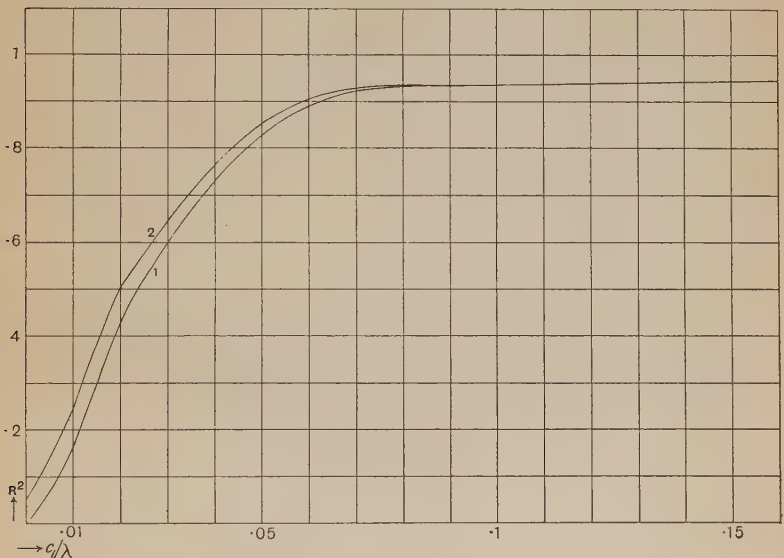

Fig. 122.

From the table or the figures it is evident that there is very little change either in the intensity or the phase of the reflected light after a thickness of one-tenth of the wave length has been reached. It appears too that over sixty per cent. of the change of phase due to a thick plate is produced by an indefinitely thin plate. It must be borne in mind, however, that the intensity of the reflected light for such thicknesses is vanishingly small, so that it would not be practicable to measure the change of phase even if such thin plates were available. And, in any case, we should

not be justified in pushing our analysis to the extreme of vanishing thinness, for when we approach molecular dimensions the

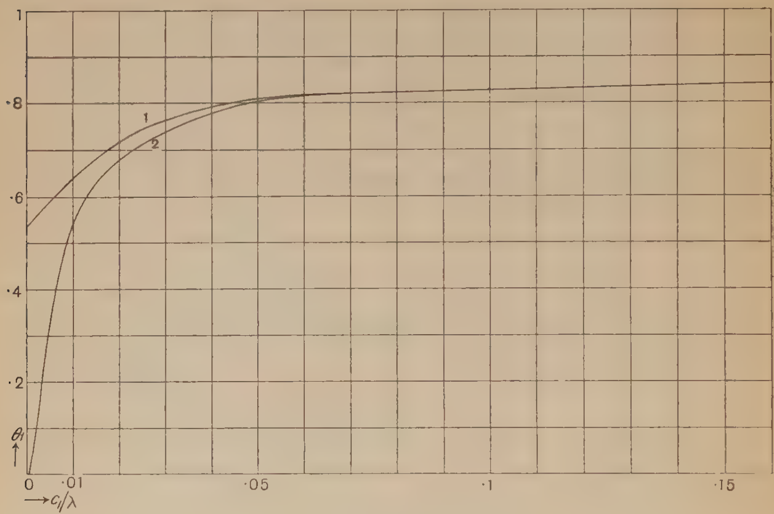

Fig. 123.

fundamental hypothesis that the granular structure of reality can be replaced for optical purposes by a continuum ceases to be tenable.

Turning to the transmitted beam we have

$$R_2 e^{i\theta_2} = q\,(1 - r^2)/(1 - q^2 r^2),$$

so that

$$R_2^{\,2} = \frac{Q\,(1 + R^4 - 2R^2 \cos\gamma)}{1 + Q^2 R^4 - 2QR^2 \cos(\psi + \gamma)},$$

and $\theta_2 = \Phi'' - \Phi - \frac{1}{2}\psi$, where Φ has the same value as before, and Φ'' is obtained from Φ by putting $Q = 1$ and $\psi = 0$. As the thickness of the plate increases Q diminishes rapidly, so that the intensity of the transmitted light diminishes rapidly to zero. Moreover as the thickness increases Φ rapidly diminishes, and $\frac{1}{2}\psi$ increases, while Φ'' is a constant. As Φ diminishes at first more rapidly than $\frac{1}{2}\psi$ increases, θ_2 begins by increasing, but it afterwards decreases and ultimately becomes negative. (It must be remembered that meanwhile Q diminishes rapidly, and the

intensity of the transmitted light becomes vanishingly small.) The values of R_2^2 and θ_2 for silver are as follows:

c_1/λ	Indef. small	·001	·005	·01	·02	·05	·1	·2
R_2^2	1	·961	·871	·742	·464	·1	·009	0
θ_2	0	·012	·064	·111	·187	·271	·279	·246

These results are represented in Figs. 124 and 125. From the first of these it appears that a silver plate whose thickness is one-tenth of the wave length is practically opaque.

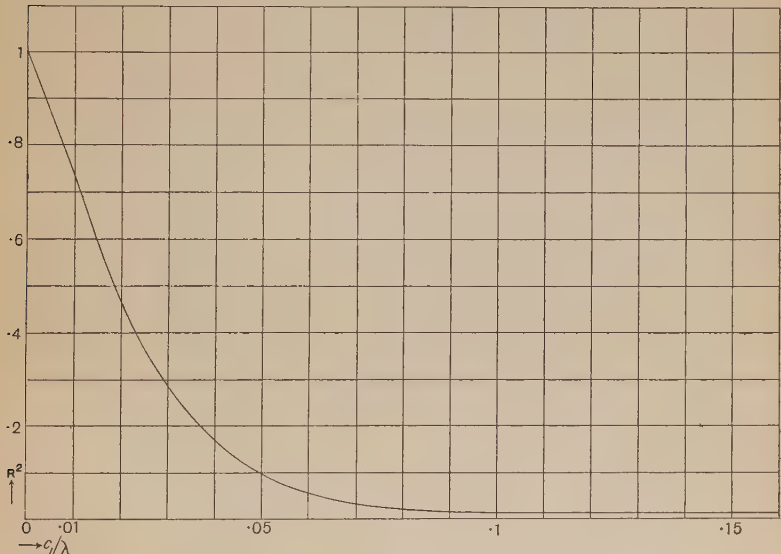

Fig. 124.

Few accurate measurements have been made on the intensity and change of phase in the case of reflection from a thin metallic plate with air surrounding it. In fact extreme tenuity can be obtained only by depositing the metal chemically or electrically on some other material such as glass. We shall therefore consider to what extent the presence of the glass modifies the character of the reflected beam. The result could be obtained by using the general formulae of p. 95, but as we have already calculated the

amplitude and phase of the reflected and transmitted beams for a metal plate surrounded by air, it is convenient to utilise these

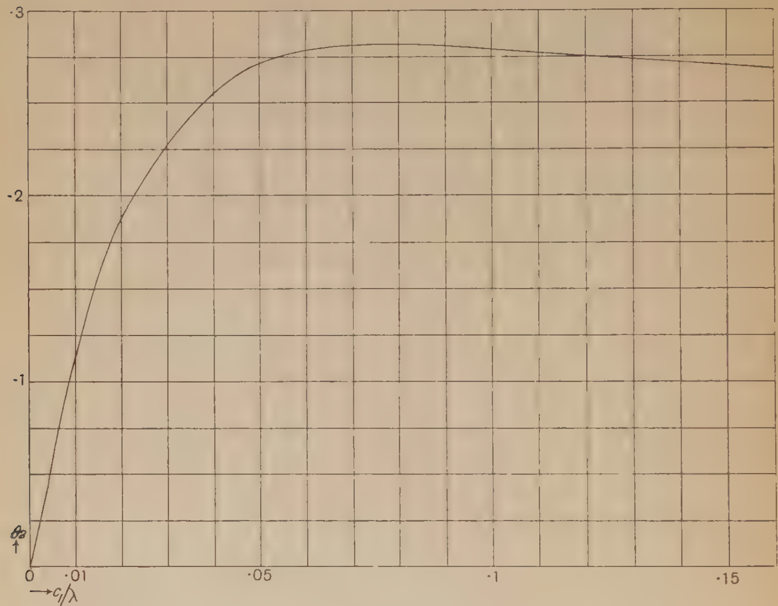

Fig. 125.

results. To enable us to do this we may suppose the metal and glass to be separated by an indefinitely thin layer of air.

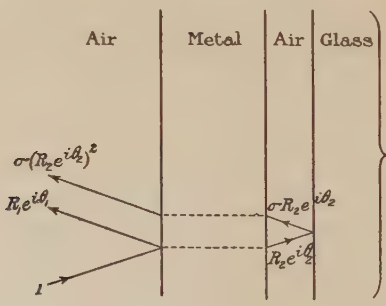

Fig. 126.

An incident wave of unit amplitude gives rise to a reflected wave $R_1e^{i\theta_1}$ and a transmitted wave $R_2e^{i\theta_2}$. The latter wave is

immediately reflected by the glass, giving a wave $\sigma R_2 e^{i\theta_2}$. This wave $\sigma R_2 e^{i\theta_2}$ is in turn transmitted through the metal plate and emerges into the air as a wave $\sigma R_2 e^{i\theta_2} . R_2 e^{i\theta_2} = Ae^{i\chi}$ say. The complete reflected wave is thus the resultant of the two components $R_1 e^{i\theta_1}$ and $Ae^{i\chi}$. If we denote this reflected wave by $R_1' e^{i\theta_1'}$ we get

$$R_1'^2 = R_1^2 + A^2 + 2R_1 A \cos(\theta_1 - \chi),$$

and
$$\tan\theta_1' = \frac{R_1 \sin\theta_1 + A \sin\chi}{R_1 \cos\theta_1 + A \cos\chi}.$$

If the incidence be direct, then σ is given by Fresnel's formula $\sigma = -(\mu - 1)/(\mu + 1)$, where μ is the refractive index of the glass. Thus

$$A = R_2^2 (\mu - 1)/(\mu + 1), \text{ and } \chi = \pi + 2\theta_2.$$

For small thicknesses R_1 is very small, while R_2 is considerable. In these circumstances R_1' is very nearly equal to A and θ_1' to χ.

As the thickness increases, R_2 rapidly diminishes, so that R_1' and θ_1' approximate more and more closely to R_1 and θ_1. Thus, except for very thin plates, the results will be the same as if the metal were surrounded by air instead of being deposited on glass. With very thin plates, however, the two cases will be quite different. Taking $\mu = 1{\cdot}54$, we obtain the following values for $R_1'^2$ and θ_1'.

c_1/λ	Indef. small	·001	·005	·01	·02	·05	·1	·2	∞
$R_1'^2$	·045	·042	·107	·241	·501	·853	·941	·952	·952
θ_1'	0	·095	·390	·536	·680	·801	·828	·831	·831

The curves marked 2 in Figs. 122 and 123 above give a graphical representation of these results. A comparison of curves 1 and 2 shows the influence of the glass. The results thus obtained are in close agreement with those found experimentally for the change of phase produced by reflection from thin films of silver deposited chemically on glass.

The same principles will enable us to calculate the ratio of the amplitudes and the difference of phase between the components of

the displacement perpendicular and parallel to the plane of incidence for reflection at any angle of incidence. We have

$$R_1 e^{i\theta_1} = r(1-q^2)/(1-q^2r^2)$$

for light polarised at right angles to the plane of incidence, and

$$R_1' e^{i\theta_1'} = r'(1-q^2)/(1-q^2r'^2)$$

for light polarised parallel to that plane. Hence

$$\frac{R_1}{R_1'} e^{i(\theta_1-\theta_1')} = \frac{r(1-q^2r'^2)}{r'(1-q^2r^2)} = \frac{R}{R'} e^{i(\theta-\theta')} \frac{1-QR'^2e^{i(2\theta'-\psi)}}{1-QR^2e^{i(2\theta-\psi)}}.$$

Thus if ϵ be the ratio of the amplitudes, ω the azimuth, and Δ the difference of phase, we have

$$\tan^2\omega = \epsilon^2 = \frac{R^2}{R'^2} \frac{1+Q^2R'^4-2QR'^2\cos(\psi+\gamma')}{1+Q^2R^4-2QR^2\cos(\psi+\gamma)}$$

and $\Delta = \theta'-\theta+\Phi-\Phi'$, where $\tan\Phi = \dfrac{QR^2\sin(\psi+\gamma)}{1-QR^2\cos(\psi+\gamma)}$, and Φ' is given by a similar formula.

If the metal is deposited on glass, we must replace

$$r(1-q^2)/(1-q^2r^2) \text{ by } (r-q^2\rho)/(1-q^2r\rho),$$

where r is the same as before, and ρ is the corresponding quantity when a ray of light goes from glass to metal instead of from air to metal. Thus in calculating ρ from the formula for r we must replace M by M/μ and ϕ by ϕ_0, where $\sin\phi = \mu\sin\phi_0$ and μ is the coefficient of refraction of the glass. Putting $\rho = Pe^{i\sigma}$ we have

$$\frac{R_1}{R_1'} e^{i(\theta_1-\theta_1')} = \frac{r-q^2\rho}{r'-q^2\rho'} \cdot \frac{1-q^2r'\rho'}{1-q^2r\rho} = \frac{r-q^2Pe^{i\sigma}}{r'-q^2P'e^{i\sigma'}} \cdot \frac{1-q^2r'P'e^{i\sigma'}}{1-q^2rPe^{i\sigma}}.$$

Hence

$$\tan^2\omega = \epsilon^2 = \frac{R^2+Q^2P^2-2QRP\cos(\sigma-\theta-\psi)}{R'^2+Q^2P'^2-2QR'P'\cos(\sigma'-\theta'-\psi)} \times \frac{1+Q^2R'^2P'^2-2QR'P'\cos(\theta'+\sigma'-\psi)}{1+Q^2RP-2QRP\cos(\theta+\sigma-\psi)},$$

and $\Delta = \chi'-\chi+\Phi-\Phi'$, where

$$\tan\chi = \frac{R\sin\theta - QP\sin(\sigma-\psi)}{R\cos\theta - QP\cos(\sigma-\psi)},$$

and

$$\tan\Phi = \frac{-QRP\sin(\theta+\sigma-\psi)}{1-QRP\cos(\theta+\sigma-\psi)}.$$

The following table gives the values of Δ for different incidences derived from these formulae for the case of silver deposited on glass of refractive index 1·54.

c_1/λ	$\phi=20°$	40°	60°	65°	70°	75°
·001	·044	·183	·466	·573	·699	·857
·005	·030	·119	·313	·397	·496	·629
·01	·025	·101	·273	·348	·445	·574
·02	·021	·090	·249	·320	·416	·542
·05	·020	·088	·240	·306	·392	·509
·1	·021	·091	·240	·302	·382	·486
·2	·021	·091	·241	·302	·380	·483

The corresponding values of the azimuth (ω) are as follows:

c_1/λ	$\phi=20°$	40°	60°	65°	70°	75°
·001	44° 47′	44° 25′	43° 20′	42° 37′	41° 29′	40° 13′
·005	43° 42′	39° 51′	32° 6′	29° 5′	26° 3′	22° 46′
·01	43° 28′	37° 37′	28° 22′	25° 50′	22° 40′	19° 54′
·02	43° 21′	38° 38′	31° 19′	29° 34′	28° 15′	26° 41′
·05	44° 29′	42° 59′	40° 28′	40° 1′	40° 9′	41° 22′
·1	44° 51′	44° 27′	43° 44′	43° 22′	43° 23′	43° 24′
·2	44° 54′	44° 36′	44° 5′	43° 23′	43° 46′	43° 41′

Fig. 127 represents Δ as a function of ϕ for different thicknesses. The curve marked 0 corresponds to an indefinitely thin film of silver on the glass, the other curves to the different thicknesses mentioned in the table above. The values of the Principal Incidence for each thickness may be obtained from the figure, and the corresponding azimuth, which is the Principal Azimuth, may then be found from a similar graphical representation of the

azimuth ω. In this way we find the following values of the Principal Incidence and the Principal Azimuth.

c_1/λ	·001	·005	·01	·02	·05	·1	·2
P. I.	61° 35′	70° 12′	72°	73° 25′	74° 45′	75° 40′	75° 48′
P. A.	43° 15′	25° 40′	21° 20′	27° 10′	41° 15′	43° 30′	43° 50′

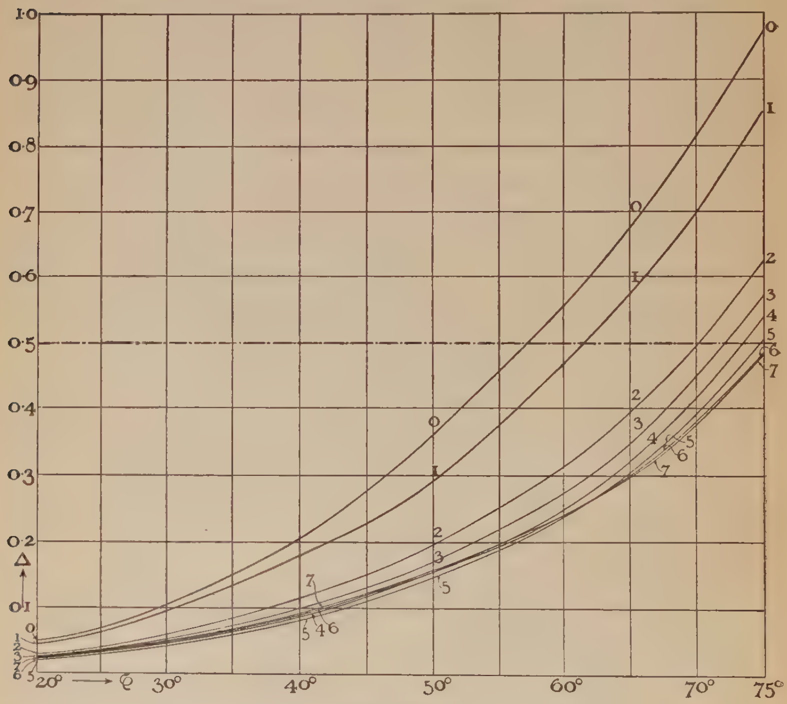

Fig. 127.

The polarising angle of the glass is 57°, and we thus see that we may pass from vitreous reflection to metallic reflection through all the intermediate states by the aid of layers of silver gradually increasing in thickness. One of the most striking features of these results is the very rapid change in the Principal Incidence produced by a slight thickening of the silver film when the film is very thin. A film of only one-thousandth of the wave length increases the Principal Incidence from 57° to 61° 35′. This indicates how con-

siderable would be the influence of a surface layer of any kind on the position of the Principal Incidence, and serves to explain the observation of all careful experimenters that the Principal Incidence alters with different conditions of the reflecting surface. It is also in good agreement with an experiment by Mascart on the subject. A silver film was deposited on glass by electrolysis. The film was so thin that its existence could not have been suspected without previous knowledge. It was impossible to estimate the thickness exactly, but it was certainly not greater than five-thousandths of a wave length. Even with this thin film the properties of the light reflected from the glass were considerably modified, and the Principal Incidence had increased from 57° to 59° 45′. Conroy made various experiments with silver films on glass. The films were of such a thickness that c_1/λ varied from ·069 to ·153, a range that lies between curves 5 and 7 of Fig. 127. One graph makes the Principal Incidence increase in this range from 74° 45′ to 75° 48′, while Conroy found an increase of 2° 4′. Although the Principal Incidence changes rapidly at first, it soon tends to a constant value, and there is little further alteration when the thickness exceeds ·00003 mm. Quincke made a large number of observations on the reflection of red light from silver films. He found that the value of the Principal Incidence increased with the thickness, tended to a constant value and changed very little when the thickness of the silver film exceeded ·00002 mm.

As the last example of these principles we shall consider the character of the beam reflected from a metallic film deposited on glass, when the condition of the surface is such that the influence of the layer of transition into the metal is appreciable. We have seen in the last chapter (p. 251) that the effect of the layer is to replace r by $r(1+pe^{i\lambda})$, where p and λ are quantities depending on the thickness and nature of the layer of transition; and the method of determining the various constants has been indicated on p. 260. We can thus calculate, by the aid of the formulae already obtained, the amplitude and phase of the light reflected from and transmitted through a metallic plate surrounded by air. The method described on p. 288 enables us to derive from these results the amplitude and phase of the reflected light when the metal is deposited on glass. The cases of light polarised parallel and perpendicular to the plane

of incidence may be dealt with separately, and the ratio of the amplitudes and the difference of phase thus obtained. On using the optical constants derived from Meslin's experiments on reflection from gold, referred to on p. 260, and taking 1·54 as the refractive index of the glass, we find the following values of the amplitude of the reflected wave, when the light is polarised at right angles to the plane of incidence.

ϕ	$c_1/\lambda=$ ·01	·02	·05	·1	·2
30°	·221	·385	·687	·854	·899
40°	·201	·362	·667	·843	·887
50°	·179	·338	·638	·812	·857
60°	·180	·316	·573	·723	·749
70°	·278	·379	·618	·743	·750
80°	·529	·599	·756	·842	·850

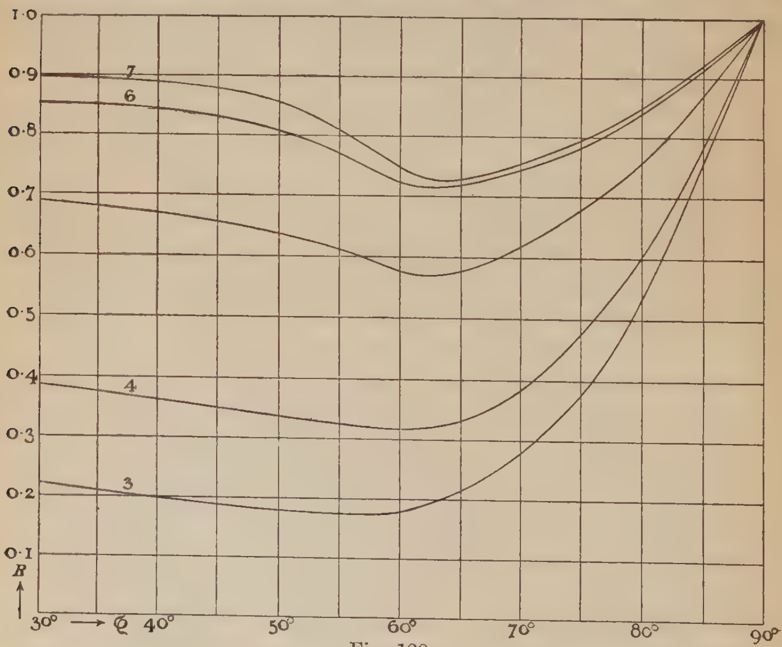

Fig. 128.

These results are represented in Fig. 128. The amplitude diminishes rapidly with the thickness. It diminishes as the incidence increases, reaches a minimum, and then increases to unity. The "quasi-polarising" angle, i.e. the angle of incidence for which the amplitude of the reflected light is least, increases slowly with the thickness of the film. For the thicknesses represented in the above table and figure this quasi-polarising angle has the values 58°, 60°, 62°, 63° and 64°.

The azimuth (ω) is given in the following table, and represented in Fig. 129.

ϕ	$c_1/\lambda = \cdot 01$	$\cdot 02$	$\cdot 05$	$\cdot 1$	$\cdot 2$
30°	38° 31′	39° 7′	42° 14′	43° 37′	44° 3′
40°	32° 59′	34° 53′	40° 22′	42° 52′	43° 22′
50°	25° 42′	30° 8′	37° 43′	41° 0′	41° 50′
60°	21° 2′	25° 40′	33° 52′	37° 50′	38° 16′
70°	24° 36′	26° 17′	34° 11′	37° 48′	37° 56′
80°	33° 49′	34° 15′	37° 58′	38° 1′	38° 58′

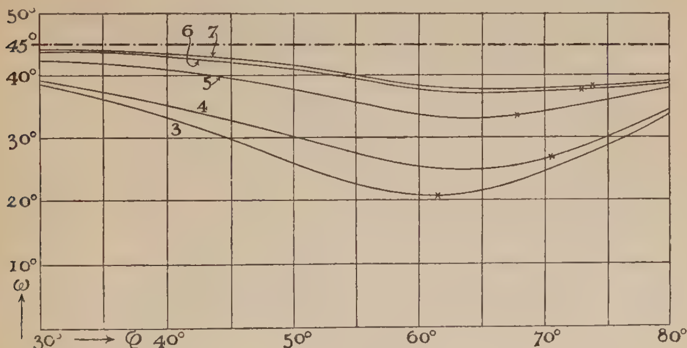

Fig. 129.

The crosses indicate the position of the Principal Azimuth, which increases through the following values: 21°, 26° 45′, 33° 30′, 37°, 37° 30′. Finally, the phase difference (Δ) has the following values, represented in Fig. 130.

ϕ	$c_1/\lambda = \cdot 01$	·02	·05	·1	·2
30°	·008	·033	·046	·055	·058
40°	·028	·086	·105	·104	·105
50°	·079	·190	·214	·190	·178
60°	·483	·317	·335	·280	·273
70°	·887	·493	·525	·440	·428
80°	·990	·755	·805	·695	·665

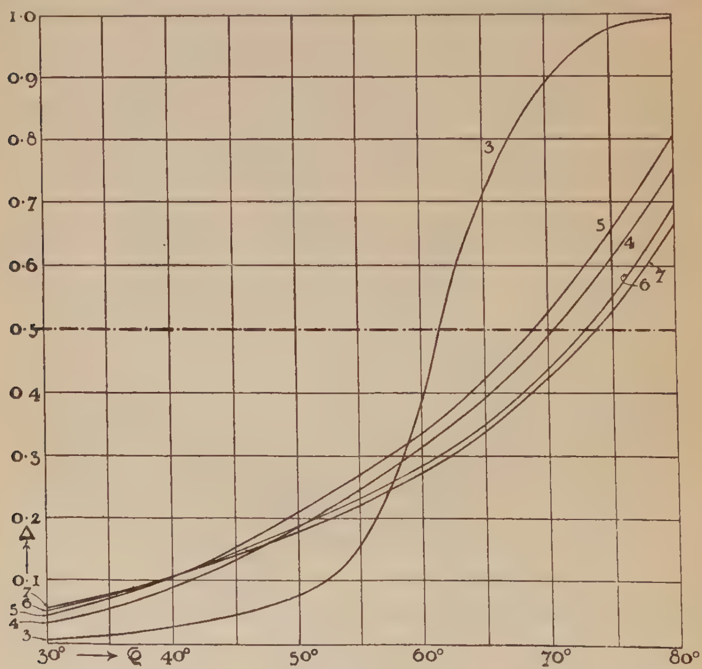

Fig. 130.

A comparison with Fig. 127 will show some marked differences. In the present case the difference of phase increases with the thickness for angles of incidence less than 40°. For larger angles, however, the law is completely different. When ϕ is greater than 60°, the difference of phase at first diminishes rapidly with the

thickness, then increases for a while, and afterwards diminishes again till it reaches the value appropriate to a thick plate. The curve (3) corresponding to $c_1/\lambda = \cdot 01$ has a point of inflexion, and cuts all the other curves. As might be expected, its form is similar to that of the curve representing the difference of phase in the case of reflection from glass or any other transparent medium when account is taken of the layer of transition. The Principal Incidence, instead of increasing steadily with the thickness, as is the case with gold alone, increases rapidly at first, diminishes for a time and then increases again. Its values for the thicknesses here discussed are 61° 30′, 70° 30′, 67° 45′, 73° and 74°.

The graphical representation of Meslin's results in Fig. 131 will

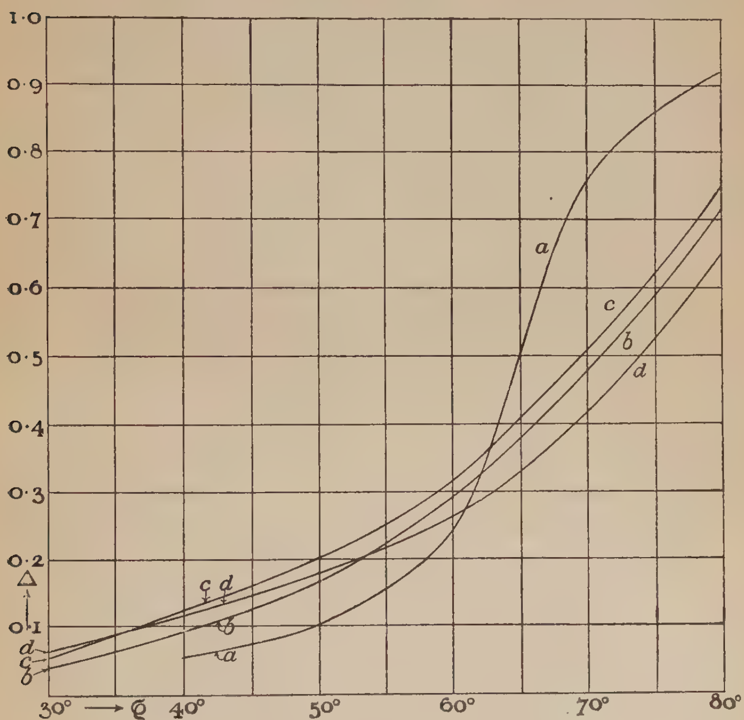

Fig. 131.

facilitate a comparison between theory and experiment, and shows that the agreement is a close one.

CHAPTER XI.

PROPAGATION OF LIGHT IN ABSORBING CRYSTALLINE MEDIA.

WE have seen in Chapter IX. that optical problems connected with absorbing media are, from the mathematical point of view, formally identical with the corresponding problems for transparent bodies, the only difference being that constants that are real in the one case are complex in the other. If then we pass from isotropic to crystalline media we should expect to derive everything as before from the fundamental Principle of Least Action, on taking the potential energy function of the form $W = \int \Psi d\tau$, where

$$2\Psi = A_{11}f^2 + A_{22}g^2 + A_{33}h^2 + 2A_{23}gh + 2A_{31}hf + 2A_{12}fg,$$

the coefficients being *complex* quantities. If we separate the real from the imaginary terms by putting $A_{11} = a_{11} + ia_{11}'$, and similarly for the other coefficients, we get $\Psi = \psi + i\psi'$, where ψ and ψ' are both expressions of the second degree in f, g, h, each involving six constants. This will be the case if the coordinate axes be taken at random; but by a proper choice of axes it will always be possible to reduce either 2ψ or $2\psi'$ to the form $a^2f^2 + b^2g^2 + c^2h^2$ or $a'^2f^2 + b'^2g^2 + c'^2h^2$ respectively. As a rule, however, the principal axes of the functions ψ and ψ' will be in different directions, and where it is necessary to distinguish them the axes of ψ may be called the *displacement axes* and those of ψ' the *absorption axes*. In general, then, there will be nine constants required to specify the function Ψ, the number being reduced to six when the displacement and the absorption axes coincide.

When the potential energy function is known the Principle of Action leads immediately to the boundary conditions and the

dynamical equations. For the former it requires the continuity of

$$m(A_{31}f + A_{32}g + A_{33}h) - n(A_{21}f + A_{22}g + A_{23}h),$$

i.e. of $m\dfrac{\partial\Psi}{\partial h} - n\dfrac{\partial\Psi}{\partial g}$, and of two similar quantities, where l, m, n are the direction cosines of the normal to the interface. This indicates that the tangential components of

$$\left(\frac{\partial\Psi}{\partial f}, \frac{\partial\Psi}{\partial g}, \frac{\partial\Psi}{\partial h}\right)$$

must be continuous at an interface. The dynamical equations prove to be

$$\ddot{\xi} = \frac{\partial}{\partial z}\left(\frac{\partial\Psi}{\partial g}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\Psi}{\partial h}\right),$$

and two similar equations. If, as before, we introduce the vector (ξ_1, η_1, ζ_1) of which (ξ, η, ζ) is the curl, we obtain

$$\frac{\partial\ddot{\zeta}_1}{\partial y} - \frac{\partial\ddot{\eta}_1}{\partial z} = \ddot{\xi} = \frac{\partial}{\partial z}\left(\frac{\partial\Psi}{\partial g}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\Psi}{\partial h}\right),$$

and two similar equations. These are all satisfied by

$$\ddot{\xi}_1 = -\frac{\partial\Psi}{\partial f}, \quad \ddot{\eta}_1 = -\frac{\partial\Psi}{\partial g}, \quad \ddot{\zeta}_1 = -\frac{\partial\Psi}{\partial h}.$$

Thus the boundary conditions require the tangential components of (ξ_1, η_1, ζ_1) to be continuous at the interface. Moreover we have $(f, g, h) = (\text{curl})^2 (\xi_1, \eta_1, \zeta_1)$, so that

$$f = -\nabla^2\xi_1 + \frac{\partial}{\partial x}\left(\frac{\partial\xi_1}{\partial x} + \frac{\partial\eta_1}{\partial y} + \frac{\partial\zeta_1}{\partial z}\right),$$

and the dynamical equations may be expressed in the form

$$\ddot{f} = \nabla^2\frac{\partial\Psi}{\partial f} - \frac{\partial\theta}{\partial x},$$

and two similar equations, where

$$\theta = \frac{\partial^2\Psi}{\partial f\partial x} + \frac{\partial^2\Psi}{\partial g\partial y} + \frac{\partial^2\Psi}{\partial h\partial z}.$$

In obtaining these equations we have imposed no restrictions as to the directions of the coordinate axes, which may form any orthogonal system whatever. Before attempting to make any deductions from the dynamical equations, it is convenient to

choose the axes so as to simplify the expression for Ψ as much as possible. We have seen that the displacement and absorption axes do not in general coincide, so that we cannot put 2Ψ in the canonical form $A^2F^2+B^2G^2+C^2H^2$ by means of a real transformation. This, however, may always be done by employing coordinates (X, Y, Z) connected with the real coordinates (x, y, z) by means of the accompanying orthogonal scheme of transformation. In this scheme (l, m, n) denote complex direction cosines connected by the relations common to all orthogonal systems, viz.

	X	Y	Z
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

$$l_1^2+l_2^2+l_3^2=1, \qquad l_2l_3+m_2m_3+n_2n_3=0,$$

and so forth.

The problem of obtaining the constants A, B, C, l, m, n, is the familiar one of finding the magnitudes and directions of the principal axes of a conicoid whose general equation is

$$2\Psi=A_{11}x^2+A_{22}y^2+A_{33}z^2+2A_{23}yz+2A_{31}zx+2A_{12}xy=1.$$

We have

$$\begin{aligned}2\Psi&=A_{11}x^2+A_{22}y^2+A_{33}z^2+2A_{23}yz+2A_{31}zx+2A_{12}xy\\&=A^2X^2+B^2Y^2+C^2Z^2,\end{aligned}$$

and since the transformation is orthogonal we also have

$$2\Phi=x^2+y^2+z^2=X^2+Y^2+Z^2.$$

Hence on equating the discriminants of the two expressions for $\Psi-\lambda\Phi$, we get

$$\Delta=\begin{vmatrix}A_{11}-\lambda & A_{21} & A_{31}\\ A_{12} & A_{22}-\lambda & A_{32}\\ A_{13} & A_{23} & A_{33}-\lambda\end{vmatrix}=(A^2-\lambda)(B^2-\lambda)(C^2-\lambda)$$

so that A^2, B^2, C^2 are the roots of $\Delta=0$.

To find the direction cosines of the principal axes we make use of the fact that the normal at the end of such an axis is in the direction of the axis, so that

$$\frac{\partial\Psi}{\partial x}\Big/l_1=\frac{\partial\Psi}{\partial y}\Big/l_2=\frac{\partial\Psi}{\partial z}\Big/l_3.$$

Hence on putting x, y, z proportional to l_1, l_2, l_3 we get

$$(A_{11}-\lambda)\,l_1+A_{21}l_2+A_{31}l_3=0,$$
$$A_{12}l_1+(A_{22}-\lambda)\,l_2+A_{32}l_3=0,$$
$$A_{13}l_1+A_{23}l_2+(A_{33}-\lambda)\,l_3=0.$$

On eliminating $l_1:l_2:l_3$ from these equations we get $\Delta=0$, so that λ is equal to A^2, B^2 or C^2, and these being known the above equations combined with $l_1^2+l_2^2+l_3^2=1$, determine l_1, l_2, and l_3, and the other direction cosines may be found similarly.

Returning to the optical problem and putting

$$(F,\,G,\,H)=(\Lambda,\,\mathrm{M},\,\mathrm{N})\,Ae^{iK(Lx+My+Nz-Vt)}$$
$$=(\Lambda,\,\mathrm{M},\,\mathrm{N})\,Ae^{iK(L_1X+M_1Y+N_1Z-Vt)},$$

where $L_1=l_1L+l_2M+l_3N$, and similarly for M_1 and N_1, while Λ, M, N, L, M, N, and V may all be complex, the problem becomes formally identical with that discussed at the beginning of Chap. VI. Thus, as on p. 132, we obtain the following relations:

$$L\Lambda+M\mathrm{M}+N\mathrm{N}=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)},$$
$$\Lambda:\mathrm{M}:\mathrm{N}=L_1/(V^2-A^2):M_1/(V^2-B^2):N_1/(V^2-C^2)\ldots\text{(II)},$$
$$V^2=A^2\Lambda^2+B^2\mathrm{M}^2+C^2\mathrm{N}^2\ldots\ldots\ldots\ldots\ldots\text{(III)},$$
$$L_1^2/(V^2-A^2)+M_1^2/(V^2-B^2)+N_1^2/(V^2-C^2)=0\ldots\text{(IV)}.$$

Before entering upon any discussion of these equations it will be well to consider the meaning of the formula

$$(F,\,G,\,H)=(\Lambda,\,\mathrm{M},\,\mathrm{N})\,A^{i\theta},$$

where Λ, M, N and θ are complex. The result has been indicated on p. 25. On putting $\Lambda=\lambda+i\lambda'$, similarly for the other complex quantities, and proceeding as on p. 25, we get

$$F=Ae^{-\theta'}(\lambda\cos\theta-\lambda'\sin\theta),$$

and so for G and H. Solving any two of these equations for $\sin\theta$ and $\cos\theta$ and eliminating θ, we get an equation representing an elliptic cylinder if F, G, H be regarded as current coordinates. Further, if we have

$$l\lambda+m\mu+n\nu=0\quad\text{and}\quad l\lambda'+m\mu'+n\nu'=0,$$

then $lF+mG+nH=0$, so that $(F,\,G,\,H)$ lies on a plane. Thus

the point (F, G, H) moving on a plane section of an elliptic cylinder describes an ellipse, and formulae such as

$$(F, G, H) = (\Lambda, \mathrm{M}, \mathrm{N})\, Ae^{i\theta}$$

indicate that the polarisation is in general elliptical. Moreover we have

$$r^2 = F^2 + G^2 + H^2 = A^2 e^{-2\theta'}\left[(\lambda^2+\mu^2+\nu^2)\cos^2\theta \right.$$
$$\left. + (\lambda'^2+\mu'^2+\nu'^2)\sin^2\theta - (\lambda\lambda'+\mu\mu'+\nu\nu')\sin 2\theta\right].$$

Thus the maximum and minimum values of r^2 are proportional to

$$\lambda^2+\mu^2+\nu^2 \quad \text{and to} \quad \lambda'^2+\mu'^2+\nu'^2,$$

in other words (λ, μ, ν) and (λ', μ', ν') are proportional to the direction cosines of the major and minor axes of the ellipse.

Let us consider next the expression $Lx + My + Nz$ that occurs in the formula for (F, G, H). On putting $L = l + il'$, where l and l' are real, and making a similar substitution for M and N, we get

$$(lx + my + nz) + i\,(l'x + m'y + n'z).$$

From this it appears that unless

$$l'/l = m'/m = n'/n = -\,\kappa \text{ (say)},$$

the displacement at any moment will not be the same at all points of a wave front $lx + my + nz =$ constant. We shall limit the discussion to cases in which all points of the wave front are in a similar condition at any time. We may then put

$$L = l\,(1 - i\kappa), \quad M = m\,(1 - i\kappa), \quad N = n\,(1 - i\kappa),$$

and so have (F, G, H) in the form

$$(F, G, H) = (\Lambda, \mathrm{M}, \mathrm{N})\, Ae^{ip[t-(lx+my+nz)(1-i\kappa)/v]},$$

where l, m, n, κ and v are real. The light will thus be elliptically polarised, and the waves will be plane. A wave will move in the direction (l, m, n) with velocity v, and with an amplitude diminishing in the ratio of $e^{-p\kappa/v}$ to unity, as the wave moves through unit distance. The quantity κ/v, on which the rate of absorption depends, may be called the *index of absorption**.

Since L, M, N are proportional to l, m, n we see from the equation $L\Lambda + M\mathrm{M} + N\mathrm{N} = 0$ of p. 301 that $l\Lambda + m\mathrm{M} + n\mathrm{N} = 0$, and, on separating the real and imaginary parts, that

$$l\lambda + m\mu + n\nu = 0 \quad \text{and} \quad l\lambda' + m\mu' + n\nu' = 0.$$

* This quantity is sometimes called the coefficient of absorption.

Thus both axes of the elliptic vibration are at right angles to the wave normal, and the displacement lies entirely in the wave front.

These results have been obtained by means of a complex scheme of transformation, but some important features of the propagation may be brought into view without any such transformation. Thus let us consider a plane wave whose normal is along the axis of z, which may be in any direction desired. With the limitations that we have already imposed, we may put

$$f = \lambda A e^{ip(t-z/V)}; \quad g = \mu A e^{ip(t-z/V)}; \quad h = 0.$$

The dynamical equations yield

$$\ddot{f} = A_{11}\frac{\partial^2 f}{\partial z^2} + A_{12}\frac{\partial^2 g}{\partial z^2}; \quad \ddot{g} = A_{12}\frac{\partial^2 f}{\partial z^2} + A_{22}\frac{\partial^2 g}{\partial z^2};$$

i.e. $\quad (A_{11} - V^2)\lambda + A_{12}\mu = 0$, and $A_{12}\lambda + (A_{22} - V^2)\mu = 0.$

On eliminating the ratio $\lambda : \mu$, we get

$$(A_{11} - V^2)(A_{22} - V^2) = A_{12}^2.$$

From this we see that there are two values of V and therefore two of v and κ, so that in any direction two waves may be propagated, the index of absorption being different for the two. Moreover, on eliminating V^2 from the above equations, we get

$$A_{12}(\lambda/\mu)^2 + (A_{22} - A_{11})(\lambda/\mu) - A_{12} = 0.$$

Hence, unless $A_{12} = 0$, we have two values of λ/μ, both of which are complex owing to the complex character of A_{11}, A_{12}, and A_{22}. The product of the two values of λ/μ is -1, so that if the roots be λ_1/μ_1 and λ_2/μ_2 we have $\lambda_1/\mu_1 = -\mu_2/\lambda_2$. From what has been said above, it appears that the fact that λ and μ are complex indicates that in each of the two waves the light is elliptically polarised; and, as we shall see, the relation $\lambda_1/\mu_1 = -\mu_2/\lambda_2$ indicates that the elliptic orbits in the two waves are similar and have their major axes at right angles. To prove this, we may take the axis of x along the major axis of one ellipse, and put

$$f_1 = A_1 \cos\alpha_1, \text{ and } g_1 = -iA_1 \sin\alpha_1,$$

omitting the factor $e^{ip(t-z/V)}$, where $\tan\alpha_1$ is the ratio of the axes of the ellipse. If the major axis of the second ellipse make an angle θ with the axis of x, we shall have similarly

$$f_2 \cos\theta + g_2 \sin\theta = A_2 \cos\alpha_2,$$

and

$$-f_2 \sin\theta + g_2 \cos\theta = -iA_2 \sin\alpha_2.$$

From these we obtain

$$\frac{\lambda_2}{\mu_2}=\frac{f_2}{g_2}=\frac{\cos\alpha_2\cos\theta+i\sin\alpha_2\sin\theta}{\cos\alpha_2\sin\theta-i\sin\alpha_2\cos\theta}=-\frac{\mu_1}{\lambda_1}=-\frac{g_1}{f_1}=\frac{i\sin\alpha_1}{\cos\alpha_1},$$

whence

$$\cos\theta\cos(\alpha_2+\alpha_1)+i\sin\theta\sin(\alpha_2-\alpha_1)=0,$$

so that we must have $\cos\theta\cos(\alpha_2+\alpha_1)=0$ as well as

$$\sin\theta\sin(\alpha_2-\alpha_1)=0.$$

These conditions require either $\theta=0$ and $\alpha_2+\alpha_1=\pi/2$, or $\theta=\pi/2$ and $\alpha_2=\alpha_1$. Both conditions express the same fact, that the two ellipses are similar and have their major axes at right angles.

We thus have an interesting generalisation of the result obtained when dealing with the propagation of light in transparent crystals. We found in that case that, in general, two waves could be propagated in any direction, and that these two waves were polarised rectilinearly, the planes of polarisation being at right angles. With absorbing crystals we also have two waves in any direction, but these are elliptically polarised and the major axes of the elliptic orbits are at right angles. It should be observed that in the special case when $A_{12}=0$, the dynamical equations yield $(A_{11}-V^2)f=0$ and $(A_{22}-V^2)g=0$, so that either $f=0$ and $V^2=A_{22}$ or $g=0$ and $V^2=A_{11}$. In this case the waves are plane polarised, the planes of polarisation being at right angles.

The equations obtained on p. 301 give the complete solution of the problem of the propagation of light in an absorbing crystalline medium. The quantities involved are all complex and the separation of the real and the imaginary terms will make each of these equations give rise to two others. When this process is carried out the formulae lose their resemblance to the simple ones of Fresnel, and become much more complicated than these. In their general form they are almost intractable, but they simplify greatly when the absorption is small. Unless this is the case the light will be so quickly absorbed that there will be no optical phenomena to observe, so that the case of weak absorption is one of special importance.

Taking the coordinate axes along the displacement axes we have

$$2\Psi=A_{11}f^2+A_{22}g^2+A_{33}h^2+2A_{23}gh+2A_{31}hf+2A_{12}fg,$$

where $A_{11} = a^2 + ia'^2$, $A_{23} = ia_{23}'$, and similarly for the other coefficients. We shall proceed to obtain an approximate solution of the general problem by supposing the absorption so weak that we may neglect squares and higher powers of the coefficients due to the absorption, viz. a'^2, a_{23}' etc. In carrying out this process we must, as indicated above, first find the lengths and directions of the principal axes of an ellipsoid whose equation is of the form $(abcfgh \between xyz)^2 = 1$, where f, g, h are small quantities of the first order. If f, g, h were zero the equation of the ellipsoid would be referred to the principal axes, so that these axes (X, Y, Z) must lie very near the coordinate axes (x, y, z). It is thus obvious geometrically that we make an error of the second order of small quantities if, in calculating the lengths of the principal axes, we take $aX^2 + bY^2 + cZ^2 = 1$ as the equation of the ellipsoid when referred to those axes. Moreover the direction cosines of the new axes will be very nearly $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, so that if we denote them by (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) as before, we shall have l_1, m_2, and n_3 equal to unity (to our order of approximation), while the other direction cosines will be small quantities of the first order. To find these quantities we have

$$(al_1 + hl_2 + gl_3)/l_1 = (hl_1 + bl_2 + fl_3)/l_2 = (gl_1 + fl_2 + cl_3)/l_3 = a,$$

with similar equations for m and n. Solving these and remembering that l_2 and l_3 are of the first order of small quantities, we get

$$l_1 = 1, \quad l_2 = h/(a-b), \quad l_3 = g/(a-c).$$

Similarly

$$m_1 = h/(b-a),\ m_2 = 1,\ m_3 = f/(b-c);\ n_1 = g/(c-a),\ n_2 = f/(c-b),\ n_3 = 1.$$

Hence in the optical problem we have

$$\begin{aligned} L_1 = l_1 L + l_2 M + l_3 N &= (1 - i\kappa)\,(ll_1 + ml_2 + nl_3) \\ &= (1 - i\kappa)\left[l + i\left\{\frac{ma_{12}'}{a^2 - b^2} + \frac{na_{31}'}{a^2 - c^2}\right\}\right], \end{aligned}$$

with similar expressions for M_1 and N_1.

Also we have

$$V^2 = v^2/(1 - i\kappa)^2 = v^2(1 + 2i\kappa),$$

to our order of approximation.

Substituting in (IV) p. 301 and separating the real from the imaginary terms, we get

$$l^2/(v^2-a^2)+m^2/(v^2-b^2)+n^2/(v^2-c^2)=0,$$

and

$$\begin{aligned}2\kappa v^2\,[l^2/(v^2-a^2)^2+m^2/(v^2-b^2)^2+n^2/(v^2-c^2)^2]\\ =a'^2l^2/(v^2-a^2)^2+b'^2m^2/(v^2-b^2)^2+c'^2n^2/(v^2-c^2)^2\\ +2a_{23}'mn/(v^2-b^2)(v^2-c^2)+2a_{31}'nl/(v^2-c^2)(v^2-a^2)\\ +2a_{12}'lm/(v^2-a^2)(v^2-b^2).\end{aligned}$$

The first of these is Fresnel's equation, and shows that the velocity of propagation in any direction is not altered by weak absorption. The second determines the index of absorption for each of the waves travelling in a given direction.

Again from (II) p. 301 we have

$$\lambda+i\lambda'=\Lambda=\frac{\kappa_1 L_1}{V^2-A^2}=\frac{\kappa_1'(l+il')}{v^2-a^2-i(a'^2-2\kappa v^2)}$$

$$=\frac{\kappa_1' l}{v^2-a^2}+il_1',$$

where l' and l_1' are small.

Hence

$$\lambda:\mu:\nu=l/(v^2-a^2):m/(v^2-b^2):n/(v^2-c^2).$$

This determines the direction of the principal axis of the elliptic orbit in a given wave, and shows that

$$(b^2-c^2)l/\lambda+(c^2-a^2)m/\mu+(a^2-b^2)n/\nu=0.$$

Further from (III) p. 301 we deduce

$$\begin{aligned}v^2(1+2i\kappa)=(a^2+ia'^2)(\lambda^2+2i\lambda\lambda')\\ +(b^2+ib'^2)(\mu^2+2i\mu\mu')+(c^2+ic'^2)(\nu^2+2i\nu\nu'),\end{aligned}$$

whence $$v^2=a^2\lambda^2+b^2\mu^2+c^2\nu^2$$

and $$2\kappa v^2=a'^2\lambda^2+b'^2\mu^2+c'^2\nu^2+2a^2\lambda\lambda'+2b^2\mu\mu'+2c^2\nu\nu'.$$

The first of these equations gives the velocity of the wave in terms of the direction of the principal axis of the elliptic orbit, and the second gives similarly the index of absorption. We may obtain a different form for $2\kappa v^2$ by eliminating l, m, n from the equations

$$\lambda:\mu:\nu=l/(v^2-a^2):m/(v^2-b^2):n/(v^2-c^2),$$

and

$$2\kappa v^2 [l^2/(v^2-a^2)^2 + m^2/(v^2-b^2)^2 + n^2/(v^2-c^2)^2]$$
$$= a'^2 l^2/(v^2-a^2)^2 + b'^2 m^2/(v^2-b^2)^2 + c'^2 n^2/(v^2-c^2)^2$$
$$+ 2a_{23}' mn/(v^2-b^2)(v^2-c^2) + 2a_{31}' nl/(v^2-c^2)(v^2-a^2)$$
$$+ 2a_{12}' lm/(v^2-a^2)(v^2-b^2).$$

This gives

$$2\kappa v^2 = a'^2\lambda^2 + b'^2\mu^2 + c'^2\nu^2 + 2a_{23}'\mu\nu + 2a_{31}'\nu\lambda + 2a_{12}'\lambda\mu.$$

Adding to these equations the equations $l\lambda + m\mu + n\nu = 0$ and $l\lambda' + m\mu' + n\nu' = 0$, which were obtained on p. 301, we have a series which give us the complete solution of the problem of propagation in weakly absorbing crystals. A comparison with the equations of Chapter VI. on propagation in transparent crystals will show that the equations not involving κ are identical in the two cases, the only difference being that (λ, μ, ν) for absorbing media no longer represents the direction of the curl of the displacement, but that of the major axis of the elliptic orbit. It follows, of course, that all the theorems previously derived from these equations apply equally to the problem at present in hand.

There is one case in which the equation

$$2\kappa v^2 [l^2/(v^2-a^2)^2 + m^2/(v^2-b^2)^2 + n^2/(v^2-c^2)^2]$$
$$= a'^2 l^2/(v^2-a^2)^2 + b'^2 m^2/(v^2-b^2)^2 + c'^2 n^2/(v^2-c^2)^2$$
$$+ 2a_{23}' mn/(v^2-b^2)(v^2-c^2) + 2a_{31}' nl/(v^2-c^2)(v^2-a^2)$$
$$+ 2a_{12}' lm/(v^2-a^2)(v^2-b^2)$$

cannot be employed to calculate the index of absorption. This occurs when the wave normal is along an optic axis, when $v^2 = b^2$, and $V^2 - B^2$ becomes a small quantity of the first order, viz.

$$i(2\kappa b^2 - b'^2).$$

Equation (IV) p. 301, written in the form

$$L_1^2(V^2-B^2)(V^2-C^2) + M_1^2(V^2-C^2)(V^2-A^2)$$
$$+ N_1^2(V^2-A^2)(V^2-B^2) = 0,$$

then yields $i(2\kappa b^2 - b'^2)[l^2(b^2-c^2) - n^2(a^2-b^2)] = 0$; but as the factor $l^2(b^2-c^2) - n^2(a^2-b^2)$ is zero, this equation cannot be employed to determine κ. It thus appears that the absorption in the direction of an optic axis is indeterminate, unless the

direction of the polarisation be known. When this is given, κ may be calculated from the formula

$$2\kappa v^2 = a'^2\lambda^2 + b'^2\mu^2 + c'^2\nu^2 + 2a_{23}'\mu\nu + 2a_{31}'\nu\lambda + 2a_{12}'\lambda\mu.$$

If natural light be incident normally on a crystal cut at right angles to an optic axis, we may replace it by two beams polarised at right angles. As the absorption for these two beams will be different, one wave may be damped while the other passes through the crystal, so that the emergent light will be polarised. The formula just quoted indicates in what way the absorption depends on the direction of the elliptic orbit (λ, μ, ν). By varying the state of the polarisation we vary the absorption, so that if natural light be transmitted through a crystal and subsequently analysed, the emergent beam will show a colour depending on the plane of analysation. This is a well known fact of experience.

For purposes of comparison with experiment it is convenient to express κ and v in terms of the angles that the wave normal makes with the optic axes. The formula for v has already been obtained (p. 142), viz. $v^2 = c^2 + (a^2 - c^2)\sin^2(\theta \pm \theta')/2$. To find the corresponding formula for κ we have to express (λ, μ, ν) in terms of θ and θ'. The direction cosines of the optic axes are l_0, m_0, n_0, where

$$l_0 = \sqrt{(a^2 - b^2)/(a^2 - c^2)} = \cos\omega/2; \quad m_0 = 0;$$
$$n_0 = \pm\sqrt{(b^2 - c^2)/(a^2 - c^2)} = \pm\sin\omega/2.$$

Using the notation and figure of p. 141, OP now representing the direction of the major axis of the elliptic orbit, we have

$$l_0\lambda + n_0\nu = \cos\psi_1 = \sin\theta\cos\chi;$$
$$l_0\lambda - n_0\nu = \cos\psi_1' = \sin\theta'\cos\chi;$$
$$l_0\lambda' + n_0\nu' = \cos\psi_2 = \sin\theta\sin\chi;$$
$$l_0\lambda' - n_0\nu' = \cos\psi_2' = -\sin\theta'\sin\chi.$$

Hence

$$\lambda = \cos\chi(\sin\theta + \sin\theta')/2l_0; \quad \nu = \cos\chi(\sin\theta - \sin\theta')/2n_0;$$
$$\mu = -\sqrt{1 - \lambda^2 - \nu^2};$$

and
$$\lambda' = \sin\chi(\sin\theta - \sin\theta')/2l_0;$$
$$\nu' = \sin\chi(\sin\theta + \sin\theta')/2n_0; \quad \mu' = -\sqrt{1 - \lambda'^2 - \nu'^2};$$

while χ is given by $\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos 2\chi$, or

$$\cos^2\chi = \frac{\sin^2(\theta+\theta')/2 - \sin^2\omega/2}{\sin\theta\sin\theta'} = \frac{\sin^2(\theta+\theta')/2 - (b^2-c^2)/(a^2-c^2)}{\sin\theta\sin\theta'}.$$

Thus all the quantities entering into the formula for κ are expressed in terms of θ and θ'.

There is a considerable simplification when the displacement and absorption axes coincide, in which case $a_{23}' = a_{31}' = a_{12}' = 0$. We then have

$$2\kappa v^2 = a'^2\lambda^2 + b'^2\mu^2 + c'^2\nu^2 = b'^2 + \lambda^2(a'^2 - b'^2) - \nu^2(b'^2 - c'^2).$$

Hence

$$2\kappa_1 v_1^2 = b'^2 + \frac{(a^2-c^2)\sin^2(\theta+\theta')/2 - (b^2-c^2)}{\sin\theta\sin\theta'}$$
$$\left[\frac{a'^2-b'^2}{a^2-b^2}\sin^2\frac{\theta+\theta'}{2}\cos^2\frac{\theta-\theta'}{2} - \frac{b'^2-c'^2}{b^2-c^2}\cos^2\frac{\theta+\theta'}{2}\sin^2\frac{\theta-\theta'}{2}\right],$$

and

$$2\kappa_2 v_2^2 = b'^2 + \frac{(b^2-c^2)-(a^2-c^2)\sin^2(\theta-\theta')/2}{\sin\theta\sin\theta'}$$
$$\left[\frac{a'^2-b'^2}{a^2-b^2}\sin^2\frac{\theta-\theta'}{2}\cos^2\frac{\theta+\theta'}{2} - \frac{b'^2-c'^2}{b^2-c^2}\cos^2\frac{\theta-\theta'}{2}\sin^2\frac{\theta+\theta'}{2}\right].$$

As a special example of the use of the formula for κ, we shall find the index of absorption when light is incident nearly at right angles to the face of a biaxal crystal cut perpendicularly to one of the optic axes. Let ON be any wave normal in the crystal near the axis OA_1, $A_1N = \theta = r$ (say) where r is small, $A_2N = \theta'$,

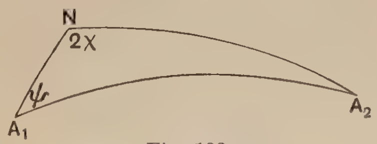

Fig. 132.

$NA_1A_2 = \psi$. If we neglect powers of r above the first we have $2\chi + \psi = 180°$; so that $\cos\chi = \sin\psi/2$ and $\sin\chi = \cos\psi/2$. On putting $1 - \cos\omega\cos\psi = 2p$ and $1 + \cos\omega\cos\psi = 2q$, we have these relations:

$$\sin\theta' = \sin\omega - r\cos\omega\cos\psi;$$
$$\sin\theta + \sin\theta' = 2(\sin\omega/2 \,.\, \cos\omega/2 + pr);$$
$$\sin\theta - \sin\theta' = 2(-\sin\omega/2 \,.\, \cos\omega/2 + qr).$$

Hence

$$\lambda = (\sin \omega/2 + pr \sec \omega/2) \sin \psi/2 ;$$
$$\lambda' = (-\sin \omega/2 + qr \sec \omega/2) \cos \psi/2 ;$$
$$\mu = -\cos \psi/2 + r \cos \psi/2 \,.\, \tan^2\psi/2 \,.\, (p \tan \omega/2 - q \cot \omega/2) ;$$
$$\mu' = -\sin \psi/2 + r \sin \psi/2 \,.\, \cot^2\psi/2 \,.\, (p \cot \omega/2 - q \tan \omega/2) ;$$
$$\nu = (-\cos \omega/2 + qr \operatorname{cosec} \omega/2) \sin \psi/2 ;$$
$$\nu' = (\cos \omega/2 + pr \operatorname{cosec} \omega/2) \cos \psi/2.$$

Hence the formula for $2\kappa v^2$ gives

$$\begin{aligned}
2\kappa_1 v_1^2 = {} & \sin^2\psi/2 \,.\, (a'^2 \sin^2 \omega/2 + c'^2 \cos^2 \omega/2) + b'^2 \cos^2 \psi/2 \\
& + 2a_{23}' \cos \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2 \\
& - 2a_{31}' \sin \omega/2 \,.\, \cos \omega/2 \,.\, \sin^2 \psi/2 - 2a_{12}' \sin \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2 \\
& + 2r \,[p (a'^2 + b'^2) \tan \omega/2 \,.\, \sin^2 \psi/2 + q (b'^2 - c'^2) \cot \omega/2 \,.\, \sin^2 \psi/2 \\
& - a_{23}' \{q \operatorname{cosec} \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2 \\
& + \cos \omega/2 \,.\, \cos \psi/2 \,.\, \tan^2 \psi/2 \,.\, (p \tan \omega/2 - q \cot \omega/2)\} \\
& + a_{31}' \cos \omega \cos \psi \sin^2 \psi/2 \\
& + a_{12}' \{\sin \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2 \,.\, \tan^2 \psi/2 \,.\, (p \tan \omega/2 - q \cot \omega/2) \\
& - p \sec \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2\}] \\
= {} & A_1 + B_1 r \;\; \text{(say)},
\end{aligned}$$

and

$$\begin{aligned}
2\kappa_2 v_2^2 = {} & \cos^2 \psi/2 \,.\, (a'^2 \sin^2 \omega/2 + c'^2 \cos^2 \omega/2) \\
& + b'^2 \sin^2 \psi/2 - 2a_{23}' \cos \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2 \\
& - 2a_{31}' \sin \omega/2 \,.\, \cos \omega/2 \,.\, \cos^2 \psi/2 + 2a_{12}' \sin \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2 \\
& - 2r \,[q (a'^2 - b'^2) \tan \omega/2 \,.\, \cos^2 \psi/2 + p (b'^2 - c'^2) \cot \omega/2 \,.\, \cos^2 \psi/2 \\
& + a_{23}' \{p \operatorname{cosec} \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2 \\
& + \cos \omega/2 \,.\, \sin \psi/2 \,.\, \cot^2 \psi/2 \,.\, (q \tan \omega/2 - p \cot \omega/2)\} \\
& - a_{31}' \cos \omega \cos \psi \cos^2 \psi/2 \\
& + a_{12}' \{\sin \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2 \,.\, \cot^2 \psi/2 \,.\, (p \cot \omega/2 - q \tan \omega/2) \\
& + q \sec \omega/2 \,.\, \sin \psi/2 \,.\, \cos \psi/2\}] \\
= {} & A_2 + B_2 r.
\end{aligned}$$

These determine κ_1 and κ_2, since, as was seen in Chapter VIII., we may put $v_1 = v_2 = b$, to our order of approximation.

The formulae just obtained cannot be employed when the wave normal coincides with an optic axis, for then ψ becomes indetermi-

nate. In this case, if the major axis of the elliptic orbit be parallel to the plane of the optic axes we have

$$\lambda = \sin \omega/2, \quad \mu = 0, \quad \nu = -\cos \omega/2,$$

so that

$$2\kappa_1' b^2 = 2\kappa_1 v_1^2 = a'^2 \sin^2 \omega/2 + c'^2 \cos^2 \omega/2 - 2a_{31}' \sin \omega/2 \cos \omega/2.$$

If, however, the major axis be perpendicular to the plane of the optic axes, we have $\lambda = \nu = 0$, and

$$2\kappa_2' b^2 = 2\kappa_2 v_2^2 = b'^2.$$

Since B and r are small quantities we may, as an approximation, neglect their product, and we then obtain κ_1 and κ_2 in terms of κ_1' and κ_2' thus:

$$\kappa_1 = \kappa_1' \sin^2 \psi/2 + \kappa_2' \cos^2 \psi/2 + p' \sin \psi,$$

and
$$\kappa_2 = \kappa_1' \cos^2 \psi/2 + \kappa_2' \sin^2 \psi/2 - p' \sin \psi,$$

where
$$2b^2 p' = a_{23}' \cos \omega/2 - a_{12}' \sin \omega/2.$$

We must next inquire into the modifications of the formulae when the crystal is uniaxal. The results cannot be derived by simply putting $b = c$ in the formulae already obtained, for when this condition is satisfied some of the steps in the process adopted require alteration. The conicoid $(A_{11} A_{22} A_{33} A_{23} A_{31} A_{12} \between xyz)^2 = 1$ is now very nearly an ellipsoid of revolution about the axis of x, so that although one principal axis must lie near the axis of x, the others need not be near the axes of y and z.

As, however, the first axis lies near the axis of x, and the other two are nearly at right angles thereto, we see that l_2, l_3, m_1 and n_1 are small quantities of the first order. Putting

$$2a_{23}'/(b'^2 - c'^2) = \tan 2\gamma,$$

we find the directions of the axes given by the following equations:

$$l_1 = 1; \quad l_2 = i a_{12}'/(a^2 - c^2); \quad l_3 = i a_{31}'/(a^2 - c^2);$$

$$m_1 = -i(a_{12}' \cos \gamma + a_{31}' \sin \gamma)/(a^2 - c^2); \quad m_2 = \cos \gamma; \quad m_3 = \sin \gamma;$$

$$n_1 = i(a_{12}' \sin \gamma - a_{31}' \cos \gamma)/(a^2 - c^2); \quad n_2 = -\sin \gamma; \quad n_3 = \cos \gamma.$$

Whence

$$\begin{aligned} L_1 : M_1 : N_1 = {} & [l + i(m a_{12}' + n a_{13}')/(a^2 - c^2)] \\ & : [m \cos \gamma + n \sin \gamma - il(a_{12}' \cos \gamma + a_{31}' \sin \gamma)/(a^2 - c^2)] \\ & : [-m \sin \gamma + n \cos \gamma + il(a_{12}' \sin \gamma - a_{31}' \cos \gamma)/(a^2 - c^2)]. \end{aligned}$$

Substituting in (IV), p. 301, the real part of the equation gives us

$$l^2/(v^2-a^2)+(m^2+n^2)/(v^2-c^2)=0,$$

which is Fresnel's law as before. From this we get

$$v_1^2=l^2c^2+a^2(1-l^2), \text{ and } v_2^2=c^2,$$

the first referring to the extraordinary and the second to the ordinary wave.

Taking the ordinary wave first, we see that V^2-B^2 and V^2-C^2 are small quantities of the first order, so that equation (IV), which is equivalent to

$$L_1^2(V^2-B^2)(V^2-C^2)+M_1^2(V^2-C^2)(V^2-A^2)$$
$$+N_1^2(V^2-A^2)(V^2-B^2)=0,$$

reduces, to our order of approximation, to

$$M_1^2(V^2-C^2)+N_1^2(V^2-B^2)=0.$$

This gives

$$(m\cos\gamma+n\sin\gamma)^2(2\kappa c^2-c'^2)+(-m\sin\gamma+n\cos\gamma)^2(2\kappa c^2-b'^2)=0,$$

whence

$$2\kappa_2c^2(m^2+n^2)=b'^2(-m\sin\gamma+n\cos\gamma)^2+c'^2(m\cos\gamma+n\sin\gamma)^2,$$

so that on putting

$$l=\cos\theta,\quad m=\sin\theta\cos\phi,\quad\text{and}\quad n=\sin\theta\sin\phi,$$

we get $$2\kappa_2c^2=b'^2\sin^2(\phi-\gamma)+c'^2\cos^2(\phi-\gamma).$$

For the extraordinary wave we get, similarly,

$$2\kappa v^2\left[\frac{l^2}{(v^2-a^2)^2}+\frac{m^2+n^2}{(v^2-c^2)^2}\right]=\frac{2l(ma_{12}'+na_{31}')}{(v^2-a^2)(v^2-c^2)}$$
$$+\frac{l^2a'^2}{(v^2-a^2)^2}+\frac{b'^2(m\cos\gamma+n\sin\gamma)^2+c'^2(-m\sin\gamma+n\cos\gamma)^2}{(v^2-c^2)^2},$$

whence, putting in the value of v^2, we find

$$2\kappa_1[l^2c^2+a^2(1-l^2)]=-2l(ma_{12}'+na_{31}')+a'^2(1-l^2)$$
$$+\{b'^2(m\cos\gamma+n\sin\gamma)^2+c'^2(-m\sin\gamma+n\cos\gamma)^2\}\,l^2/(1-l^2),$$

i.e. $$2\kappa_1[c^2\cos^2\theta+a^2\sin^2\theta]=-\sin 2\theta\,(a_{12}'\cos\phi+a_{31}'\sin\phi)$$
$$+a'^2\sin^2\theta+\cos^2\theta\,[b'^2\cos^2(\phi-\gamma)+c'^2\sin^2(\phi-\gamma)].$$

In the special case where there is complete symmetry about the optic axis we have $b'=c'$, $a_{12}'=a_{31}'=0$, and thus get

$$2\kappa_2c^2=c'^2,\quad\text{and}\quad 2\kappa_1[c^2\cos^2\theta+a^2\sin^2\theta]=c'^2\cos^2\theta+a'^2\sin^2\theta,$$

where θ is the angle that the wave normal makes with the optic axis. If κ_0 be the value of κ_1 when $\theta = 0$, we have

$$\frac{\kappa_1}{\kappa_0} = \frac{1 + (a'/c')^2 \tan^2 \theta}{1 + (a/c)^2 \tan^2 \theta}.$$

On putting in the values of a/c and a'/c' found experimentally for tourmaline, we are enabled to compare the results of theory and experiment. This is set out in the following table, and shown graphically in Fig. 133.

θ...	0	10°	20°	30°	40°	50°	60°	70°	80°	90°
κ_1/κ_0 (theory)	1	·990	·960	·914	·857	·798	·742	·697	·667	·657
κ_1/κ_0 (exp.)	·998	·987	·943	·898	·834	·814	·734	·719	·661	·657
Difference	+ ·002	+ ·003	+ ·017	+ ·016	+ ·023	− ·016	+ ·008	− ·012	+ ·006	0

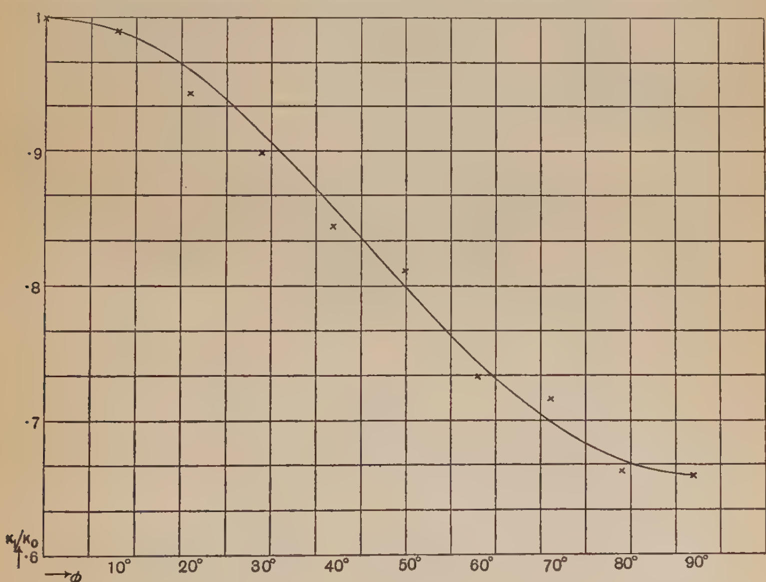

Fig. 133.

We are now in a position to see in what way the index of absorption $\nu = \kappa/v$ varies with the direction of propagation of the wave (l, m, n). From Fresnel's equation it appears that there are,

in general, two velocities for a given direction, and from the equations for κ we see that κ has a different value for the two velocities. Hence there are two values of ν for each direction. If then through any point we draw radii vectores whose directions represent the directions of propagation of the wave, and whose lengths are equal to ν, the surface so formed will consist of two sheets. It may be called the *Absorptive Index Surface*, on the analogy of the Index Surface of Chapter V. Its equation in polar coordinates (r, θ, ϕ) is obtained by putting $l = \cos\theta$, $m = \sin\theta\cos\phi$, $n = \sin\theta\sin\phi$, and eliminating κ and v from the equations $\kappa/v = r$, and those already obtained involving v^2 and $2\kappa v^2$. In the case of uniaxal crystals we thus get

$$2rc^3 = b'^2 \sin^2(\phi - \gamma) + c'^2 \cos^2(\phi - \gamma)$$

for one sheet, and

$$\begin{aligned} 2r\,[c^2\cos^2\theta + a^2\sin^2\theta]^{3/2} = &- \sin 2\theta\,(a_{12}'\cos\phi + a_{31}'\sin\phi) \\ &+ a'^2\sin^2\theta + \cos^2\theta\,[b'^2\cos^2(\phi-\gamma) + c'^2\sin^2(\phi-\gamma)] \end{aligned}$$

for the other sheet. In the special case of complete symmetry about the optic axis, these reduce to $2rc^3 = c'^2$ and

$$2r\,[c^2\cos^2\theta + a^2\sin^2\theta]^{3/2} = c'^2\cos^2\theta + a'^2\sin^2\theta$$

respectively. Fig. 134 represents a quadrant of the section of this surface by a plane through the optic axis, drawn to scale when $c = 0{\cdot}611$, $a = 0{\cdot}615$, and $(a'/c')^2 = 0{\cdot}665$, the constants being those found experimentally for tourmaline.

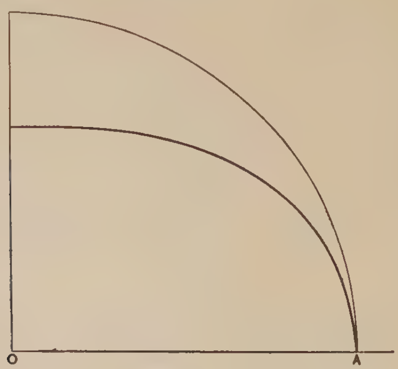

Fig. 134.

In general the two sheets of the Absorptive Index Surface will intersect, and the lines joining the origin to the points of intersection will form a cone, which may be called the *cone of equal absorption*, from the fact that the absorption will be the same for all waves whose normals are generators of this cone. We have seen that for a wave whose normal coincides with the optic axis the absorptive index cannot be derived from the equation

$$\begin{aligned}
2\kappa v^2 [l^2/(v^2-a^2)^2 + m^2/(v^2-b^2)^2 + n^2/(v^2-c^2)^2] \\
= a'^2 l^2/(v^2-a^2)^2 + b'^2 m^2/(v^2-b^2)^2 + c'^2 n^2/(v^2-c^2)^2 \\
+ 2a_{23}'\, mn/(v^2-b^2)(v^2-c^2) + 2a_{31}'\, nl/(v^2-c^2)(v^2-a^2) \\
+ 2a_{12}'\, lm/(v^2-a^2)(v^2-b^2),
\end{aligned}$$

but that when the direction of the displacement is known its various values are determined by the equation

$$2\kappa v^2 = a'^2\lambda^2 + b'^2\mu^2 + c'^2\nu^2 + 2a_{23}'\mu\nu + 2a_{31}'\nu\lambda + 2a_{12}'\lambda\mu.$$

From this it appears that the optic axes are not in general generators of the cone of equal absorption. The equation of this cone is obtained by giving equal roots to the equation of the absorptive index surface, regarded as an equation in r, and by putting $l : m : n = x : y : z$. In this way we obtain for a uniaxal crystal the equation

$$\begin{aligned}
[b'^2(-y\sin\gamma + z\cos\gamma)^2 + c'^2(y\cos\gamma + z\sin\gamma)^2]\,[c^2x^2 + a^2(y^2+z^2)]^{3/2} \\
= c^3(x^2+y^2+z^2)^{1/2}\,[(y^2+z^2)\,\{a'^2(y^2+z^2) - 2a_{12}'xy - 2a_{31}'xz\} \\
+ \{b'^2(xy\cos\gamma + xz\sin\gamma)^2 + c'^2(-xy\sin\gamma + xz\cos\gamma)^2\}].
\end{aligned}$$

In the case of complete symmetry the cone becomes a circular one with its axis along the optic axis, and its semi-vertical angle θ given by the equation

$$2q^3\sin^2\theta = p^2 - 3q \pm (p-q)^{3/2}(p+3q)^{1/2},$$

where $\quad p = (a'^2 - c'^2)/a'^2$, and $q = (a^2-c^2)/a^2$,

and, of course, unless θ be real the cone of equal absorption does not exist.

The phenomena analogous to those discussed in Chapter VII. when dealing with absorbing crystals may be investigated by the aid of the formulae obtained in this chapter. We have to remember that light incident on an absorbing plate gives rise to two waves within the crystal, and that these waves have different velocities and are differently absorbed. The difference in the velocities

leads to a difference of phase in the two waves on emergence, and as the velocities are given, to our order of approximation, by the same formulae as when the crystal is transparent, the change of phase is the same as in the cases discussed in Chapter VII. The absorption produces a change of amplitude, which can be calculated for each wave from the formulae of this chapter. On passing through the crystalline plate the amplitude of a wave is diminished in the ratio of $e^{-\sigma}$ to unity, where $\sigma = pl\nu$, p/π being the frequency, ν the index of absorption, and l the length of the wave normal within the crystal. Hence an incident wave represented by Ae^{ipt} will, on emergence, be represented by $Ae^{-\sigma}e^{i(pt-\Delta)}$, where Δ is the change of phase. Since $\sigma = pl\nu = 2\pi l\kappa/\lambda$, where λ is the wave length, we see that σ will be large and consequently $e^{-\sigma}$ very small, when the thickness of the plate is large, in all cases in which κ is appreciable. If, however, for any direction κ be very small, $e^{-\sigma}$ may be appreciable even when the plate is not very thin.

Let us suppose, as in Chapter VII., that we have a polariser and analyser whose principal planes OP and OA make angles α and β respectively with Ox, the plane of polarisation of the quicker wave within the crystalline plate. For brevity let c_1, s_1, c_2, s_2 denote $\cos\alpha$, $\sin\alpha$, $\cos\beta$, and $\sin\beta$ respectively. On resolving an incident vibration Ae^{ipt} along Ox and Oy respectively, Oy being at right angles to Ox, and omitting the factor e^{ipt}, we get Ac_1 and As_1. After passing through the plate these become $Ac_1e^{-\sigma_1}e^{-i\Delta_1}$ and $As_1e^{-i\sigma_2}e^{-i\Delta_2}$ respectively, where σ_1 and σ_2, Δ_1 and Δ_2 refer to the two waves, the ellipticity of the polarisation within the crystal being neglected, as this is very slight in cases to which the formulae will be applied. On resolving along OA we get

$$A\left[c_1c_2e^{-\sigma_1}e^{-i\Delta_1} + s_1s_2e^{-\sigma_2}e^{i\Delta_2}\right],$$

so that the intensity I is given by

$$I = A^2\left[c_1^2c_2^2e^{-2\sigma_1} + s_1^2s_2^2e^{-2\sigma_2} + 2s_1s_2c_1c_2e^{-(\sigma_1+\sigma_2)}\cos\Delta\right],$$

where $\Delta = \Delta_1 - \Delta_2$ is the difference of phase between the two waves on emergence. If the light used be not homogeneous we must add the intensities due to each wave length, as was done in Chapter VII.

Some interesting results are obtained with absorbing crystals by using unpolarised light. In this way we may replace the

incident light by two equal beams polarised at right angles, and then the above equation for the intensity gives

$$2I = A^2 [c_2^2 e^{-2\sigma_1} + s_2^2 e^{-2\sigma_2}].$$

If, moreover, there be no analyser, we get

$$2I = A^2 [e^{-2\sigma_1} + e^{-2\sigma_2}].$$

We shall consider first the case of a biaxal crystal cut at right angles to an optic axis, for which the formulae giving κ_1 and κ_2 have been explained on p. 310. The planes of polarisation of the two waves within the crystal bisect the interior and exterior angles A_1NA_2 (Fig. 132), so that with the notation of p. 309 we may take $\psi/2$ as the angle between the plane of polarisation of one of the waves and the plane of the optic axes. If then the polariser and the analyser be crossed the equation for I becomes

$$4I = A^2 \sin^2 (2\alpha_1 - \psi) [e^{-2\sigma_1} + e^{-2\sigma_2} - 2e^{-(\sigma_1 + \sigma_2)} \cos \Delta],$$

where α_1 is the angle that the principal plane of the polariser makes with the plane of the optic axes. Along the optic axis itself ψ becomes indeterminate, and we must then proceed as on p. 311 by resolving the incident light into two streams polarised at right angles. These two streams traverse the crystal at the same speed, so that there is no difference of phase between them, and the intensity is I', where

$$4I' = \sin^2 2\alpha_1 [e^{-\sigma_1'} - e^{-\sigma_2'}]^2, \text{ and } \sigma' = 2\pi l\kappa'/\lambda,$$

κ_1' and κ_2' having the values assigned to them on p. 311. Since σ_1' and σ_2' are different, we see that I' does not vanish unless $\sin 2\alpha_1 = 0$, i.e. unless $\alpha_1 = 0$ or $\pi/2$. For any other azimuth of the polariser, the extremity of the optic axis is not black.

In the more general case the equation for I shows that the intensity vanishes when $\psi = 2\alpha_1$, so that there is a black line $\psi = 2\alpha_1$ across the field of view, as in the case of transparent crystals, with the difference just noted that this black line is interrupted by a comparatively bright spot at the optic axis, except in the special cases when $\alpha_1 = 0$ or $\pi/2$.

I also vanishes with the factor $e^{-2\sigma_1} + e^{-2\sigma_2} - 2e^{-(\sigma_1 + \sigma_2)} \cos \Delta$. This cannot be zero unless $\sigma_1 = \sigma_2$ and $\Delta = 2n\pi$. The condition $\sigma_1 = \sigma_2$ involves $\kappa_1 = \kappa_2$, which from the formulae on p. 311 leads to $\tan \psi = (\kappa_1' - \kappa_2')/p'$, i.e. to $\psi = \pm \pi/2$ in the special case when

the displacement and absorption axes coincide. In this azimuth there will be black spots at distances from the optic axis given by

$$\Delta = 2n\pi, \text{ i.e. } r = 2n\lambda b^2/\epsilon C = 2n\lambda b^3/\epsilon (a^2 - c^2) \sin \omega,$$

where ϵ is the thickness of the crystalline plate (see p. 209).

To investigate the fluctuations of the intensity as r varies in any given azimuth we have to consider the variation of the factor

$$F = e^{-2\sigma_1} + e^{-2\sigma_2} - 2e^{-(\sigma_1+\sigma_2)} \cos \Delta.$$

All the quantities σ_1, σ_2, and Δ are functions of r, σ_1 and σ_2 being given by the formulae on p. 311, and Δ by the formula

$$\Delta = (\pi\epsilon/\lambda b^3)(a^2 - c^2) r \sin \omega \doteq (\pi\epsilon/\lambda b^2) Cr.$$

The intensity is stationary when $\partial I/\partial r$ is zero, i.e. when $\partial F/\partial r$ is zero. This gives

$$B_1 e^{-2\sigma_1} + B_2 e^{-2\sigma_2} - (B_1 + B_2) e^{-(\sigma_1+\sigma_2)} \cos \Delta - Ce^{-(\sigma_1+\sigma_2)} \sin \Delta = 0,$$

i.e. $$B_1 e^{-2(\sigma_1' \sin^2 \psi/2 + \sigma_2' \cos^2 \psi/2 + q' \sin \psi)} + B_2 e^{-2(\sigma_1' \cos^2 \psi/2 + \sigma_2' \sin^2 \psi/2 - q' \sin \psi)}$$
$$+ e^{-(\sigma_1' + \sigma_2')} [(B_1 + B_2) \cos \Delta + C \sin \Delta] = 0,$$

where $q' = p' \,.\, 2\pi\epsilon/\lambda$.

This equation, combined with the formulae for B_1, B_2, and C already obtained, gives r as a function of ψ and so determines the form of the rings. As either $e^{-\sigma_1'}$ or $e^{-\sigma_2'}$ will be small if there be any appreciable absorption, the factor $e^{-(\sigma_1'+\sigma_2')}$, which is associated with the terms $\sin \Delta$ and $\cos \Delta$, will be small. Thus if there be much absorption the terms involving Δ and therefore r in the formula for $\partial I/\partial r$ will be negligible, and the rings will consequently be very indistinct.

The rings are not equally bright or dark throughout their length, as the intensity varies with the azimuth. For a given value of r the intensity is stationary when $\partial I/\partial \psi$ vanishes, i.e. when

$$\sin (2\alpha_1 - \psi) [\sin (2\alpha_1 - \psi) \partial F/\partial \psi - 2F \cos (2\alpha_1 - \psi)] = 0.$$

The factor $\sin (2\alpha_1 - \psi)$ equated to zero gives $\psi = 2\alpha_1$, the black line already referred to. Also we have

$$F = e^{-2(\sigma_1' \sin^2 \psi/2 + \sigma_2' \cos^2 \psi/2 + q' \sin \psi)}$$
$$+ e^{-2(\sigma_1' \cos^2 \psi/2 + \sigma_2' \sin^2 \psi/2 - q' \sin \psi)} - 2e^{-(\sigma_1'+\sigma_2')} \cos \Delta.$$

Moreover

$$\partial \Delta/\partial \psi = (\pi\epsilon/\lambda b^3)(a^2 - c^2) r^2 \cos \omega \sin \psi,$$

and this is small owing to the factor r^2. Hence as $e^{-(\sigma_1'+\sigma_2')}$ is very small we may neglect the variation of the factor $e^{-(\sigma_1'+\sigma_2')} \cos \Delta$ in finding $\partial F/\partial \psi$. Thus $\partial I/\partial \psi$ vanishes when

$$\sin(2\alpha_1 - \psi)[(\sigma_1' - \sigma_2') \sin \psi + 2q' \cos \psi](e^{-2\sigma_1} - e^{-2\sigma_2})$$
$$+ 2\cos(2\alpha_1 - \psi)[e^{-2\sigma_1} + e^{-2\sigma_2} - 2e^{-(\sigma_1+\sigma_2)} \cos \Delta] = 0.$$

This equation is very nearly satisfied when $\sigma_1 = \sigma_2 = (\sigma_1' + \sigma_2')/2$, for then the first term vanishes and the second term becomes $8e^{-(\sigma_1'+\sigma_2')} \cos(2\alpha_1 - \psi) \sin^2 \Delta/2$, which is very small owing to the presence of the factor $e^{-(\sigma_1'+\sigma_2')}$. We have seen (p. 317) that $\tan \psi = (\kappa_1' - \kappa_2')/p'$ when $\sigma_1 = \sigma_2$, and that consequently $\psi = \pm \pi/2$ in the special case of symmetry when $p' = 0$. In this special case we see also that another approximate solution of the equation

$$\sin(2\alpha_1 - \psi)[(\sigma_1' - \sigma_2') \sin \psi + 2q' \cos \psi](e^{-2\sigma_1} - e^{-2\sigma_2})$$
$$+ 2\cos(2\alpha_1 - \psi)[e^{-2\sigma_1} + e^{-2\sigma_2} - 2e^{-(\sigma_1+\sigma_2)} \cos \Delta] = 0$$

is $\psi = 0$ or π, for when ψ has either of these values the first term of the equation vanishes and the second reduces to

$$\pm 2\cos 2\alpha_1 [e^{-2\sigma_1'} + e^{-2\sigma_2'} - 2e^{-(\sigma_1'+\sigma_2')} \cos \Delta],$$

which is small unless the plate is very thin or the absorption negligible. It has been seen that the azimuths $\psi = \pm \pi/2$ make I a minimum, so that as maxima and minima must occur alternately we should expect, and can easily verify, that the azimuths $\psi = 0$ and π correspond to maximum values of the intensity. Thus there is a bright line in the plane of the optic axes and a dark one at right angles thereto, in addition to the black line in the azimuth $\psi = 2\alpha_1$.

If the polariser and analyser be parallel, instead of being crossed, the formula for I gives

$$I = A^2[e^{-2\sigma_1} \cos^4(\alpha_1 - \psi/2) + e^{-2\sigma_2} \sin^4(\alpha_1 - \psi/2)$$
$$+ (1/2)\, e^{-(\sigma_1+\sigma_2)} \cos \Delta \sin^2(2\alpha_1 - \psi)]$$

except in the direction of the optic axis itself when the intensity is

$$I' = A^2[\cos^2 \alpha_1\, e^{-\sigma_1'} + \sin^2 \alpha_1\, e^{-\sigma_2'}]^2.$$

The discussion of these formulae may be conducted on exactly the

same lines as before. If the plate be of sufficient thickness we may neglect the term $e^{-(\sigma_1'+\sigma_2')}$, and we then have

$$\frac{1}{A^2}\frac{\partial I}{\partial\psi}=\sin(2\alpha_1-\psi)\left[e^{-2\sigma_1}\cos(\alpha_1-\psi/2)-e^{-2\sigma_2}\sin(\alpha_1-\psi/2)\right]$$
$$-2\left[e^{-2\sigma_1}\frac{d\sigma_1}{d\psi}+e^{-2\sigma_2}\frac{d\sigma_2}{d\psi}\right]$$
$$=\sin(2\alpha_1-\psi)\left[e^{-2\sigma_1}\cos(\alpha_1-\psi/2)-e^{-2\sigma_2}\sin(\alpha_1-\psi/2)\right]$$
$$-\left[e^{-2\sigma_1}-e^{-2\sigma_2}\right]\left[(\sigma_1'-\sigma_2')\sin\psi+2q'\cos\psi\right].$$

If $\sigma_1=\sigma_2=(\sigma_1'+\sigma_2')/2$, the second term vanishes, and the first is very small owing to the factor $e^{-(\sigma_1'+\sigma_2')}$. Also if $q'=0$ and $\psi=0$ or π, the second term again vanishes and the first, which is always small, vanishes completely when $\alpha_1=0$. In the case where $q'=0$, we have $\sigma_1=\sigma_2$ when $\psi=\pm\pi/2$ and then, in the immediate neighbourhood of the optic axis where $\Delta=0$, we have

$$I=A^2\left[e^{-\sigma_1}\cos^2(\alpha_1-\psi/2)+e^{-\sigma_2}\sin^2(\alpha_1-\psi/2)\right]^2=A^2e^{-(\sigma_1'+\sigma_2')}.$$

Moreover, if $\alpha_1=0$, the formula for I' gives $I'=A^2e^{-2\sigma_2'}$, so that $I/I'=e^{-(\sigma_1'-\sigma_2')}$, and I is greater or less than I' according as $e^{-\sigma_1'}$ or $e^{-\sigma_2'}$ is negligible, i.e. according to the type of the crystal employed. If $\alpha_1=\pi/2$, the relations between I and I' are reversed.

If unpolarised light be used, the intensity is given by the formula

$$2I=A^2\left[\cos^2(\alpha_2-\psi/2)\,e^{-2\sigma_1}+\sin^2(\alpha_2-\psi/2)\,e^{-2\sigma_2}\right],$$

where α_2 is the angle that the principal plane of the analyser makes with the plane of the optic axes; whereas along an optic axis itself we have

$$2I'=A^2\left[\cos^2\alpha_2\,.\,e^{-2\sigma_2'}+\sin^2\alpha_2\,.\,e^{-2\sigma_1'}\right].$$

When $\alpha_2=0$, the formulae give

$$2I=A^2\left[\cos^2\psi/2\,.\,e^{-2\sigma_1}+\sin^2\psi/2\,.\,e^{-2\sigma_2}\right],$$

and
$$2I'=A^2e^{-2\sigma_2'}.$$

Hence

$$\frac{1}{A^2}\frac{\partial I}{\partial\psi}=\left[\cos^2\psi/2\,.\,e^{-2\sigma_1}-\sin^2\psi/2\,.\,e^{-2\sigma_2}\right]$$
$$\left[(\sigma_1'-\sigma_2')\sin\psi+2q'\cos\psi\right]-\tfrac{1}{4}\sin\psi\left(e^{-2\sigma_1}-e^{-2\sigma_2}\right).$$

Thus when $q'=0$, $\partial I/\partial\psi$ vanishes when $\psi=0$ or π, and is very small when $\sigma_1=\sigma_2$, i.e. when $\psi=\pm\pi/2$. In the former case

($\psi = 0$ or π) we have $I = I'$, and in the latter case ($\psi = \pm\,\pi/2$) we have $I/I' = e^{-(\sigma_1' - \sigma_2')}$. Hence with crystals belonging to the type for which $\sigma_1' > \sigma_2'$ there is a dark line across the field of view perpendicular to the plane of the optic axes, this line being interrupted by a bright spot at the centre; while with crystals of the other type for which $\sigma_1' < \sigma_2'$ the dark line lies in the plane of the optic axes and is continuous.

When $\alpha_2 = \pi/2$ we have

$$2I = A^2 [\sin^2 \psi/2 \,.\, e^{-2\sigma_1} + \cos^2 \psi/2 \,.\, e^{-2\sigma_2}],$$

and $2I' = A^2 e^{-2\sigma_1'}$. In this case, in the azimuths $\psi = 0$ or π we have $I = I'$, and in the azimuths $\psi = \pm\,\pi/2$ we have

$$I/I' = e^{-(\sigma_2' - \sigma_1')}.$$

Hence, if $\sigma_1' > \sigma_2'$, a continuous dark line crosses the field in the plane of the optic axes, while with crystals of the other type the dark line is perpendicular to that plane and is interrupted by a bright spot at the centre.

Lastly, if there be no analyser we have $2I = A^2 [e^{-2\sigma_1} + e^{-2\sigma_2}]$, and $2I' = A^2 [e^{-2\sigma_1'} + e^{-2\sigma_2'}]$. Hence

$$\partial I/\partial \psi = A^2 [e^{-2\sigma_2} - e^{-2\sigma_1}] [(\sigma_1' - \sigma_2') \sin \psi + 2q' \cos \psi],$$

so that I is stationary when $\sigma_1 = \sigma_2$, and also when

$$\tan \psi = 2q'/(\sigma_2' - \sigma_1') = 2p'/(\kappa_2' - \kappa_1').$$

In the case of symmetry when $p' = 0$, the former corresponds to $\psi = \pm\,\pi/2$ and the latter to $\psi = 0$ or π, so that we have a dark line perpendicular to the plane of the optic axes with a bright spot at the centre.

As the final application of these formulae we shall take the case of a uniaxal plate cut at right angles to the optic axis. The displacements for the two waves are in the plane of incidence and at right angles thereto, the former corresponding to the ordinary and the latter to the extraordinary wave. If the polariser and analyser be crossed the formula for the intensity gives

$$4I = \sin^2 2\phi \, [e^{-2\sigma_1} + e^{-2\sigma_2} - 2 \cos \Delta \,.\, e^{-(\sigma_1 + \sigma_2)}],$$

where ϕ is the angle that the principal plane of the polariser makes with the plane of polarisation of the quicker wave. Along the optic axis, we have $\sigma_1 = \sigma_2$ and $\Delta = 0$, so that $I' = 0$. The intensity vanishes when $\sin 2\phi = 0$, i.e. when $\phi = 0$ or $\pi/2$, so that there is a

black cross in the field of view with its arms parallel to the principal planes of the polariser and analyser.

We have

$$\frac{\partial I}{\partial r} = -\frac{A^2}{2}\sin^2 2\phi\left[e^{-2\sigma_1}\frac{d\sigma_1}{dr} + e^{-2\sigma_2}\frac{d\sigma_2}{dr} - e^{-(\sigma_1+\sigma_2)}\left\{\cos\Delta\left(\frac{d\sigma_1}{dr}+\frac{d\sigma_2}{dr}\right) + \sin\Delta\frac{d\Delta}{dr}\right\}\right].$$

On putting $d = 1$ in the formula of p. 209, we have

$$\Delta = (\pi\epsilon/\lambda c)(a^2 - c^2) r^2.$$

Also from the formulae of p. 312 we get $\sigma_2 = (\pi\epsilon/\lambda)(c'/c)^2$, and

$$\begin{aligned}\sigma_1 &= (\pi\epsilon/\lambda)(c'^2\cos^2\theta + a'^2\sin^2\theta)/(c^2\cos^2\theta + a^2\sin^2\theta)\\ &= (\pi\epsilon/\lambda)[c'^2 + (a'^2 - c'^2)r^2]/c^2\\ &= \sigma_2[1 + r^2(a'^2 - c'^2)/c'^2].\end{aligned}$$

Hence

$$\frac{1}{A^2}\frac{\partial I}{\partial r} = -(\pi\epsilon/\lambda c^2)\, r\sin^2 2\phi \,.\, e^{-(\sigma_1+\sigma_2)} [c(a^2-c^2)\sin\Delta - (a'^2 - c'^2)\cos\Delta - e^{-r^2\sigma_2(a'^2-c'^2)/c'^2}].$$

Owing to the smallness of the factor $a'^2 - c'^2$, this yields very approximately

$$\lambda c\frac{\partial I}{\partial r} = -\pi\epsilon A^2(a^2 - c^2)\, r e^{-(\sigma_1+\sigma_2)}\sin^2 2\phi\sin\Delta,$$

so that I is a maximum or minimum when $\Delta = n\pi$. There will thus be a series of bright and dark concentric circles in the field of view, their radii being given by the same law as that which holds for transparent crystals. Owing, however, to the presence of the factor $e^{-(\sigma_1+\sigma_2)}$ in the expression for $\partial I/\partial r$, the variation of the intensity will be very slight if the absorption be at all large, so that the rings will be very indistinct. When the absorption or the thickness of the plate is considerable, we have I very nearly proportional to $(e^{-2\sigma_1} + e^{-2\sigma_2})\sin^2 2\phi$, i.e. to $e^{-2\sigma_2}[1 + e^{-r^2\sigma_2(a'^2-c'^2)/c'^2}]\sin^2 2\phi$. The term $e^{-r^2\sigma_2(a'^2-c'^2)/c'^2}$ will be very small for crystals belonging to the type in which a' is considerable and c' very small, while it will be large for crystals of the other type in which a' is very small and c' considerable. Hence with crystals of the first class the field away from the black cross will be dark, and with those of the other class it will be bright.

If the polariser and analyser be parallel, the formula for the intensity gives

$$I = A^2 [\cos^4\phi \,.\, e^{-2\sigma_1} + \sin^4\phi \,.\, e^{-2\sigma_2} + (1/2)\sin^2 2\phi \cos\Delta \,.\, e^{-(\sigma_1+\sigma_2)}],$$

and

$$\partial I/\partial\phi = 2A^2 \sin 2\phi\,[\sin^2\phi \,.\, e^{-2\sigma_2} - \cos^2\phi \,.\, e^{-2\sigma_1} + \cos 2\phi \cos\Delta \,.\, e^{-(\sigma_1+\sigma_2)}].$$

Thus I is a maximum or a minimum when $\phi = 0$ or $\pi/2$. In the former case ($\phi = 0$) we have $I_1 = A^2 e^{-2\sigma_1} = A^2 e^{-2\sigma_2 [1 + r^2 (a'^2 - c'^2)/c'^2]}$, and in the latter case ($\phi = \pi/2$) we have $I_2 = A^2 e^{-2\sigma_2}$, so that

$$I_1/I_2 = e^{-2r^2\sigma_2 (a'^2 - c'^2)/c'^2} = e^{-(2\pi\epsilon/\lambda)\, r^2 (a'^2 - c'^2)/c'^2}.$$

At the end of the optic axis we have $I = A^2 e^{-2\sigma_2} = I_2$. Hence with crystals for which a' is large compared with c', we have a bright line perpendicular to the principal plane of the polariser and a dark line at right angles thereto, this dark line being interrupted by a bright spot at the end of the optic axis. On passing to crystals of the opposite type for which c' is large compared with a', we must interchange bright and dark in the description of these lines.

When unpolarised light is employed, we have

$$2I = A^2 [\cos^2\phi \,.\, e^{-2\sigma_1} + \sin^2\phi \,.\, e^{-2\sigma_2}],$$

where ϕ is the angle that the analyser makes with the plane of polarisation of the quicker wave, and $2I' = A^2 e^{-2\sigma_2}$. From these equations we get $2\partial I/\partial\phi = A^2 \sin 2\phi\,[e^{-2\sigma_2} - e^{-2\sigma_1}]$, so that I is stationary when $\phi = 0$ or $\pi/2$, and the bright and dark lines are the same as those in the case last discussed.

Finally, if there be neither polariser nor analyser we have

$$2I = A^2 [e^{-2\sigma_1} + e^{-2\sigma_2}], \text{ and } I' = A^2 e^{-2\sigma_2}.$$

Everything is independent of ϕ, so that there are no bright or dark lines crossing the field. The intensity varies slightly with r; but if the absorption be appreciable the rings are very indistinct, and the appearance presented is that of a bright or a dark field according to the type of the crystal, the bright field having a dark spot and the dark field a bright spot at the centre.

All these conclusions relative to the behaviour of thin crystalline plates that have slight absorptive power are in thorough agreement with the results of experiment.

INDEX.

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